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On the decomposition numbers of the Hecke algebra of type D_n when n is even

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ABSTRACT

Let $n \geq 4$ be an even integer. Let K be a field with $\text{char } K \neq 2$ and q an invertible element in K such that $\prod_{i=1}^{n-1} (1 + q^i) \neq 0$. In this paper, we study the decomposition numbers over K of the Iwahori–Hecke algebra $\mathcal{H}_q(D_n)$ of type D_n . We obtain some equalities which relate its decomposition numbers with certain Schur elements and the decomposition numbers of various Iwahori–Hecke algebras of type A with the same parameter q . When $\text{char } K = 0$, this completely determine all of its decomposition numbers. The main tools we used are the Morita equivalence theorem established in [J. Hu, A Morita equivalence theorem for Hecke algebra $\mathcal{H}_q(D_n)$ when n is even, Manuscripta Math. 108 (2002) 409–430] and certain twining character formulae of Weyl modules over a tensor product of two q -Schur algebras.

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1. Introduction

Let n be a natural number. Let K be a field and q, Q two invertible elements in K . Let W_n be the Weyl group of type A_{n-1} or of type B_n . Let $\mathcal{H}(W_n)$ be the Iwahori–Hecke algebra of W_n with parameter q if $W_n = W(A_{n-1})$; or with parameters q, Q if $W_n = W(B_n)$. The modular representation theory of $\mathcal{H}(W_n)$ has been well studied in the papers [3,7–9,13] and [32]. In fact, most of the results of the modular representation theory of these algebras have been generalized to a more general class of algebras—the cyclotomic Hecke algebras of type $G(r, 1, n)$, where $r \in \mathbb{N}$. The latter was now fairly well understood by the work of [1,2,4,12] and [14].

This paper is concerned with the Iwahori–Hecke algebra $\mathcal{H}_q(D_n)$ of type D_n . By definition, $\mathcal{H}_q(D_n)$ is the associative unital K -algebra with generators T_u, T_1, \dots, T_{n-1} subject to the following relations

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$$\begin{aligned}
(T_u + 1)(T_u - q) &= 0, \\
(T_i + 1)(T_i - q) &= 0, \quad \text{for } 1 \leq i \leq n-1, \\
T_u T_2 T_u &= T_2 T_u T_2, \quad T_u T_1 = T_1 T_u, \\
T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1}, \quad \text{for } 1 \leq i \leq n-2, \\
T_i T_j &= T_j T_i, \quad \text{for } 1 \leq i < j-1 \leq n-2, \\
T_u T_i &= T_i T_u, \quad \text{for } 2 < i < n.
\end{aligned}$$

The algebra $\mathcal{H}_q(D_n)$ can be embedded into a Hecke algebra $\mathcal{H}_q(B_n)$ of type B_n with parameters $\{q, 1\}$ as a “normal” subalgebra. Namely, let $\mathcal{H}_q(B_n)$ be the associative unital K -algebra with generators T_0, T_1, \dots, T_{n-1} subject to the following relations

$$\begin{aligned}
(T_0 + 1)(T_0 - 1) &= 0, \\
T_0 T_1 T_0 T_1 &= T_1 T_0 T_1 T_0, \\
(T_i + 1)(T_i - q) &= 0, \quad \text{for } 1 \leq i \leq n-1, \\
T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1}, \quad \text{for } 1 \leq i \leq n-2, \\
T_i T_j &= T_j T_i, \quad \text{for } 0 \leq i < j-1 \leq n-2.
\end{aligned}$$

Then the map ι which sends T_u to $T_0 T_1 T_0$, and T_i to T_i (for each integer i with $1 \leq i \leq n-1$) can be uniquely extended to an injection of K -algebras. Throughout this paper, we shall always identify $\mathcal{H}_q(D_n)$ with the subalgebra $\iota(\mathcal{H}_q(D_n))$ using this embedding ι .

Henceforth, we shall assume that the characteristic of the field K (denoted by $\text{char } K$) is not equal 2. In this case, if q is not a root of unity, then $\mathcal{H}_q(D_n)$ is semisimple. Since we are only interested in the modular (i.e., non-semisimple) case, we shall also assume that q is a root of unity in K . Let e be the smallest positive integer such that $1 + q + q^2 + \dots + q^{e-1} = 0$. If $q = 1$, then $e = \text{char } K$. The modular representation theory of $\mathcal{H}_q(D_n)$ over K was studied in a number of papers [16,19,20,22,28,33]. The algebra $\mathcal{H}_q(D_n)$ is a special case of a more general class of algebra—the cyclotomic Hecke algebras of type $G(r, p, n)$. The latter was studied in [21,23–26] and [27]. The papers [19,20] and [33] studied the restriction to $\mathcal{H}_q(D_n)$ of simple $\mathcal{H}_q(B_n)$ -modules using the combinatorics of Kleshchev bipartitions; while the papers [16] and [28] studied the simple $\mathcal{H}_q(D_n)$ -modules with the aim of constructing the so-called “canonical basic set.” In both approaches, simple $\mathcal{H}_q(D_n)$ -modules have been classified but using different parameterizations.

One of the major open problems in the modular representation theory of Hecke algebras is the determination of their decomposition numbers. In the case of type A and type B (or more generally, of type $G(r, 1, n)$), thanks to the work of [1] and [30], the decomposition numbers can be computed by the evaluation at 1 of some alternating sum of certain parabolic affine Kazhdan–Lusztig polynomials when $\text{char } K = 0$ and $q \neq 1$. It is natural to ask what will happen in the type D case. In [33], it was proved that if n is odd and $f_n(q) := \prod_{i=1}^{n-1} (1 + q^i) \neq 0$ in K , then

$$\mathcal{H}_q(D_n) \stackrel{\text{Morita}}{\sim} \bigoplus_{a=(n+1)/2}^n \mathcal{H}_q(\mathfrak{S}_{(a,n-a)}),$$

where $\mathfrak{S}_{(a,n-a)} := \mathfrak{S}_{\{1,\dots,a\}} \times \mathfrak{S}_{\{a+1,\dots,n\}}$ is the parabolic subgroup of the symmetric group \mathfrak{S}_n on $\{1, 2, \dots, n\}$, $\mathcal{H}_q(\mathfrak{S}_{(a,n-a)})$ is the parabolic subalgebra of $\mathcal{H}_q(\mathfrak{S}_n)$ corresponding to $\mathfrak{S}_{(a,n-a)}$. In this case, if $S_{\underline{\lambda}}$ and $D_{\underline{\mu}}$ denote the dual Specht module and simple module of $\mathcal{H}_q(B_n)$ labelled by the bipartition $\underline{\lambda} = (\lambda^{(1)}, \lambda^{(2)})$ and the e -restricted bipartition $\underline{\mu} = (\mu^{(1)}, \mu^{(2)})$ respectively, then we have the following equality of decomposition numbers:

$$[S_{\hat{\lambda}} \downarrow \mathcal{H}_q(D_n) : D_{\hat{\mu}} \downarrow \mathcal{H}_q(D_n)] = [S_{\lambda^{(1)}} : D_{\mu^{(1)}}] [\widehat{S}_{\lambda^{(2)}} : \widehat{D}_{\mu^{(2)}}],$$

where $S_{\lambda^{(1)}}$, $D_{\mu^{(1)}}$ (resp., $\widehat{S}_{\lambda^{(2)}}$, $\widehat{D}_{\mu^{(2)}}$) denote the dual Specht module and simple module of $\mathcal{H}_q(\mathfrak{S}_a)$ (resp., of $\mathcal{H}_q(\mathfrak{S}_{\{a+1, \dots, n\}})$). Hence computing the decomposition numbers of $\mathcal{H}_q(D_n)$ can be reduced to computing the decomposition numbers of $\mathcal{H}_q(\mathfrak{S}_{(a, n-a)})$ where $(n+1)/2 \leq a \leq n$.

In [19], it was proved that if n is even and $f_n(q) \neq 0$ in K , then

$$\mathcal{H}_q(D_n) \stackrel{\text{Morita}}{\sim} A(n/2) \oplus \bigoplus_{a=n/2+1}^n \mathcal{H}_q(\mathfrak{S}_{(a, n-a)}), \quad (1.1)$$

where $A(n/2)$ is the K -subalgebra of $\mathcal{H}_q(\mathfrak{S}_n)$ generated by $\mathcal{H}_q(\mathfrak{S}_{(n/2, n/2)})$ and an invertible element $h(n/2) \in \mathcal{H}_q(\mathfrak{S}_n)$ (see [19, Definition 1.5] for definition of $h(n/2)$). Therefore, in this case, computing the decomposition numbers of $\mathcal{H}_q(D_n)$ can be reduced to computing the decomposition numbers of $\mathcal{H}_q(\mathfrak{S}_{(a, n-a)})$ where $n/2 + 1 \leq a \leq n$ and the decomposition numbers of the algebra $A(n/2)$.

The purpose of this article is to determine the decomposition numbers for the algebra $A(n/2)$. The main results of this paper provide some explicit formulae which determine these decomposition numbers of $\mathcal{H}_q(D_n)$ in terms of the decomposition numbers of $\mathcal{H}_q(\mathfrak{S}_{n/2})$ and certain Schur elements, see Theorem 2.5, Lemma 2.7 (where it is only assumed $\text{char } K \neq 2$), Theorem 2.8 (in the case where $\text{char } K = 0$) and Theorem 2.9. Note that these results are also valid in the case where $q = 1$.

The paper is organized as follows. In Section 2 we shall first recall some known results about the modular representation theory of the Hecke algebra of type D_n when n is even in the separated case. Then we shall state our main theorems. In Section 3 we lift the construction of the algebra $A(n/2)$ to the level of q -Schur algebras and introduce a certain covering $\tilde{A}(n/2)$ of $A(n/2)$. Using Schur functor, we transfer the original problem of computing the decomposition numbers of $A(n/2)$ to the corresponding problem for $\tilde{A}(n/2)$. In Section 4, we explicitly compute the Laurent polynomial $f_{\hat{\lambda}}(v)$ introduced in [19, Lemma 3.2] in terms of the Schur elements of the Hecke algebra $\mathcal{H}_q(B_n)$ and $\mathcal{H}_q(\mathfrak{S}_{n/2})$. In Section 5, by computing the twining character formula of some Weyl modules of a tensor product of two q -Schur algebras, we determine our desired decomposition numbers when K is of characteristic 0. When K is of odd characteristic, our results only give some equalities about these decomposition numbers modulo $\text{char } K$.

2. Preliminaries

From now on until the end of this paper, we assume that $n = 2m \geq 4$ is an even integer, and

$$2f_n(q) := 2 \prod_{i=1}^{n-1} (1 + q^i) \neq 0. \quad (2.1)$$

We shall refer (2.1) as separation condition and say that we are in the separated case.

Let k be a positive integer. A sequence of nonnegative integers $\lambda = (\lambda_1, \lambda_2, \dots)$ is said to be a composition of k if $\sum_{i \geq 1} \lambda_i = k$. A composition $\lambda = (\lambda_1, \lambda_2, \dots)$ of k is said to be a partition of k (denoted by $\lambda \vdash k$) if $\lambda_1 \geq \lambda_2 \geq \dots$. If λ is a composition of k , then we write $|\lambda| = k$. A bipartition of n is an ordered pair $\underline{\lambda} = (\lambda^{(1)}, \lambda^{(2)})$ of partitions such that $\lambda^{(1)}$ is a partition of a and $\lambda^{(2)}$ is a partition of $n - a$ for some integer a with $0 \leq a \leq n$. In this case, we say that $\underline{\lambda}$ is an a -bipartition of n and we also write $\underline{\lambda} \vdash n$.

A partition λ is said to be e -restricted if $0 \leq \lambda_i - \lambda_{i+1} < e$ for all i . We say that a bipartition $\underline{\lambda} = (\lambda^{(1)}, \lambda^{(2)})$ is e -restricted if both $\lambda^{(1)}$ and $\lambda^{(2)}$ are e -restricted. Recall that for each bipartition $\underline{\lambda} = (\lambda^{(1)}, \lambda^{(2)})$ of n , there is a dual Specht module¹ $S_{\hat{\lambda}}$ of $\mathcal{H}_q(B_n)$. If $\mathcal{H}_q(B_n)$ is semisimple, then the

¹ Our $S_{\hat{\lambda}}$ in this paper was denoted by \tilde{S}^{λ} in [19].

set $\{S_{\underline{\lambda}} | \underline{\lambda} = (\lambda^{(1)}, \lambda^{(2)}) \vdash n\}$ forms a complete set of pairwise non-isomorphic simple $\mathcal{H}_q(B_n)$ -modules. In general, if $\underline{\lambda}$ is e -restricted, then $S_{\underline{\lambda}}$ has a unique simple $\mathcal{H}_q(B_n)$ -head $D_{\underline{\lambda}}$, and the set

$$\{D_{\underline{\lambda}} | \underline{\lambda} = (\lambda^{(1)}, \lambda^{(2)}) \vdash n \text{ is } e\text{-restricted}\}$$

forms a complete set of pairwise non-isomorphic simple $\mathcal{H}_q(B_n)$ -modules.

Let v be an indeterminate over \mathbb{Z} . Let $\mathcal{A} := \mathbb{Z}[v, v^{-1}]$. For each bipartition $\underline{\lambda} = (\lambda^{(1)}, \lambda^{(2)})$ of n , a Laurent polynomial $f_{\underline{\lambda}}(v) \in \mathcal{A}$ was introduced in [19, Lemma 3.2]. If $\lambda^{(1)} = \lambda^{(2)}$, then it was proved in [19, Theorem 3.5] that there exist some Laurent polynomials $g_{\underline{\lambda}}(v) \in \mathcal{A}$, such that $f_{\underline{\lambda}}(v) = (g_{\underline{\lambda}}(v))^2$. Since $\text{char } K \neq 2$, we actually have two choices for such $g_{\underline{\lambda}}(v)$. Henceforth, for each bipartition $\underline{\lambda} = (\lambda^{(1)}, \lambda^{(2)})$ of n satisfying $\lambda^{(1)} = \lambda^{(2)}$, we fix one choice of such $g_{\underline{\lambda}}(v)$, and we denote it by $\sqrt{f_{\underline{\lambda}}(v)}$. Then another choice would be $-\sqrt{f_{\underline{\lambda}}(v)}$. Once this is done, we can canonically define two $\mathcal{H}_q(D_n)$ -submodules $S_{\underline{\lambda}}^+$ and $S_{\underline{\lambda}}^-$ of $S_{\underline{\lambda}} \downarrow_{\mathcal{H}_q(D_n)}$ satisfying $S_{\underline{\lambda}} \downarrow_{\mathcal{H}_q(D_n)} = S_{\underline{\lambda}}^+ \oplus S_{\underline{\lambda}}^-$ (see [21, Theorem 4.6]).

Let \mathcal{P}_n denote the set of bipartitions of n . For any $\underline{\lambda}, \underline{\mu} \in \mathcal{P}_n$, we define $\underline{\lambda} \sim \underline{\mu}$ if $\lambda^{(1)} = \mu^{(2)}$ and $\lambda^{(2)} = \mu^{(1)}$. Let τ be the K -algebra automorphism of $\mathcal{H}_q(B_n)$ which is defined on generators by $\tau(T_1) = T_0 T_1 T_0$, $\tau(T_i) = T_i$, for any $i \neq 1$. Let σ be the K -algebra automorphism of $\mathcal{H}_q(B_n)$ which is defined on generators by $\sigma(T_0) = -T_0$, $\sigma(T_i) = T_i$, for any $i \neq 0$. Clearly $\tau(\mathcal{H}_q(D_n)) = \mathcal{H}_q(D_n)$ and $\sigma \downarrow_{\mathcal{H}_q(D_n)} = \text{Id}$. With our assumption (2.1) in mind, by [21, Corollary 3.7], for each $\underline{\lambda} = (\lambda^{(1)}, \lambda^{(2)}) \in \mathcal{P}_n$,

$$S_{\underline{\lambda}} \downarrow_{\mathcal{H}_q(D_n)} \cong S_{\widehat{\underline{\lambda}}} \downarrow_{\mathcal{H}_q(D_n)}, \quad \text{where } \widehat{\underline{\lambda}} := (\lambda^{(2)}, \lambda^{(1)}).$$

Note that the inequality (2.1) is only a part of the conditions for the semisimplicity of $\mathcal{H}_q(D_n)$. The following result depends heavily on our assumption (2.1).

Lemma 2.2. (See [21].) *If $\mathcal{H}_q(D_n)$ is semisimple, then the set*

$$\{S_{\underline{\lambda}} \downarrow_{\mathcal{H}_q(D_n)}, S_{(\beta, \beta)}^+, S_{(\beta, \beta)}^- | \underline{\lambda} = (\lambda^{(1)}, \lambda^{(2)}) \in \mathcal{P}_n / \sim, \lambda^{(1)} \neq \lambda^{(2)}, \beta \vdash m\}$$

forms a complete set of pairwise non-isomorphic simple $\mathcal{H}_q(D_n)$ -modules. In general, if $\underline{\mu} \vdash n$ is e -restricted and $\mu^{(1)} \neq \mu^{(2)}$, then $D_{\underline{\mu}} \downarrow_{\mathcal{H}_q(D_n)}$ remains irreducible and it is the unique simple $\mathcal{H}_q(D_n)$ -head of $S_{\underline{\mu}} \downarrow_{\mathcal{H}_q(D_n)}$; if $\alpha \vdash m$ is e -restricted, then $S_{(\alpha, \alpha)}^+$ (resp., $S_{(\alpha, \alpha)}^-$) has a unique simple $\mathcal{H}_q(D_n)$ -head $D_{(\alpha, \alpha)}^+$ (resp., $D_{(\alpha, \alpha)}^-$). The algebra $\mathcal{H}_q(D_n)$ is split over K and the set

$$\left\{ D_{\underline{\mu}} \downarrow_{\mathcal{H}_q(D_n)}, D_{(\alpha, \alpha)}^+, D_{(\alpha, \alpha)}^- \mid \begin{array}{l} \underline{\mu} = (\mu^{(1)}, \mu^{(2)}) \in \mathcal{P}_n / \sim \text{ is } e\text{-restricted} \\ \mu^{(1)} \neq \mu^{(2)}, \alpha \vdash m \text{ is } e\text{-restricted} \end{array} \right\}$$

forms a complete set of pairwise non-isomorphic simple $\mathcal{H}_q(D_n)$ -modules.

Therefore, we can regard the following set

$$\left\{ S_{\underline{\lambda}} \downarrow_{\mathcal{H}_q(D_n)}, S_{(\beta, \beta)}^+, S_{(\beta, \beta)}^- \mid \begin{array}{l} \underline{\lambda} = (\lambda^{(1)}, \lambda^{(2)}) \in \mathcal{P}_n / \sim, \lambda^{(1)} \neq \lambda^{(2)}, \text{ and} \\ \beta \text{ is a partition of } m \end{array} \right\}$$

as the set of dual Specht modules for $\mathcal{H}_q(D_n)$ in the separated case.

We collect together some facts in the following lemma.

Lemma 2.3. (See [19, 21].) *Let β be a partition of m and α be an e -restricted partition of m . Let $\underline{\lambda}$ be a bipartition of n and $\underline{\mu}$ be an e -restricted bipartition of n . Then we have*

(2.3.1) $f_{\underline{\lambda}}(q)$ is an invertible element in K ;

(2.3.2) $(D_{\underline{\mu}})^\sigma \cong D_{\underline{\mu}}, D_{\underline{\mu}} \downarrow_{\mathcal{H}_q(D_n)} \cong D_{\underline{\mu}} \downarrow_{\mathcal{H}_q(D_n)}$;

$$(2.3.3) \quad D_{(\alpha, \alpha)} \downarrow \mathcal{H}_q(D_n) \cong D_{(\alpha, \alpha)}^+ \oplus D_{(\alpha, \alpha)}^-;$$

$$(2.3.4) \quad (S_{(\beta, \beta)}^+)^{\tau} \cong S_{(\beta, \beta)}^-, \quad (D_{(\alpha, \alpha)}^+)^{\tau} \cong D_{(\alpha, \alpha)}^-;$$

$$(2.3.5) \quad M^{\tau} \cong M \text{ for any } \mathcal{H}_q(B_n)\text{-module } M.$$

Note that except for (2.3.5), all the claims in the above lemma depend on the validity of our assumption (2.1).

Recall the Morita equivalence (1.1) proved in [19]. Let F be the resulting functor from the category of finite dimensional $\mathcal{H}_q(D_n)$ -modules to the category of finite dimensional modules over $A(m) \oplus \bigoplus_{a=m+1}^n \mathcal{H}_q(\mathfrak{S}_{(a, n-a)})$. For any $\underline{\lambda}, \underline{\mu} \in \mathcal{P}_n$ with $\underline{\mu}$ being e -restricted, we define

$$S(\underline{\lambda}) := S_{\lambda^{(1)}} \otimes \widehat{S}_{\lambda^{(2)}}, \quad D(\underline{\mu}) := D_{\mu^{(1)}} \otimes \widehat{D}_{\mu^{(2)}},$$

if $|\lambda^{(1)}| \neq |\lambda^{(2)}|$, $|\mu^{(1)}| \neq |\mu^{(2)}|$; and

$$S(\underline{\lambda}) := (S_{\lambda^{(1)}} \otimes \widehat{S}_{\lambda^{(2)}}) \uparrow_{\mathcal{H}_q(\mathfrak{S}_{(m, m)})}^{A(m)}, \quad D(\underline{\mu}) := (D_{\mu^{(1)}} \otimes \widehat{D}_{\mu^{(2)}}) \uparrow_{\mathcal{H}_q(\mathfrak{S}_{(m, m)})}^{A(m)},$$

if $|\lambda^{(1)}| = |\lambda^{(2)}| = m = |\mu^{(1)}| = |\mu^{(2)}|$.

For each integer i with $1 \leq i \leq m-1$, we define $s_i = (i, i+1)$. Then $\{s_1, s_2, \dots, s_{m-1}\}$ is the set of all the simple reflections in \mathfrak{S}_m . A word $w = s_{i_1} \cdots s_{i_k}$ for $w \in \mathfrak{S}_m$ is a reduced expression if k is minimal; in this case we say that w has length k and we write $\ell(w) = k$. Given a reduced expression $s_{i_1} \cdots s_{i_k}$ for $w \in \mathfrak{S}_m$, we write $T_w = T_{i_1} \cdots T_{i_k}$. It is well known that $\{T_w | w \in \mathfrak{S}_m\}$ forms a K -basis of $\mathcal{H}_q(\mathfrak{S}_m)$. Let β be a partition of m . Let \mathfrak{S}_{β} be the Young subgroup of \mathfrak{S}_m corresponding to β . We set

$$x_{\beta} := \sum_{w \in \mathfrak{S}_{\beta}} T_w, \quad y_{\beta} := \sum_{w \in \mathfrak{S}_{\beta}} (-q)^{-\ell(w)} T_w.$$

Let t^{β} (resp., t_{β}) be the standard β -tableau in which the numbers $1, 2, \dots, m$ appear in order along successive rows (resp., columns). Let $w_{\beta} \in \mathfrak{S}_m$ be such that $t^{\beta} w_{\beta} = t_{\beta}$. If α is an e -restricted partition of m , then we have the following direct sum decompositions of $A(m)$ -modules:

$$S(\beta, \beta) = S(\beta, \beta)_{+} \oplus S(\beta, \beta)_{-}, \quad D(\alpha, \alpha) = D(\alpha, \alpha)_{+} \oplus D(\alpha, \alpha)_{-},$$

where

$$S(\beta, \beta)_{\pm} := (\sqrt{f_{(\beta, \beta)}(q)} z'_{\beta} \otimes \widehat{z}_{\beta} \pm (z'_{\beta} \otimes \widehat{z}_{\beta}) h(m)) \mathcal{H}_q(\mathfrak{S}_{(m, m)}),$$

$$D(\alpha, \alpha)_{\pm} := (\sqrt{f_{(\alpha, \alpha)}(q)} \overline{z}_{\alpha} \otimes \widehat{z}_{\alpha} \pm (\overline{z}_{\alpha} \otimes \widehat{z}_{\alpha}) h(m)) \mathcal{H}_q(\mathfrak{S}_{(m, m)}),$$

and $z'_{\beta} := y_{\beta'} T_{w_{\beta'}} x_{\beta}$, β' is the conjugate partition of β , \widehat{z}_{β} is the similarly defined generator for the dual Specht module \widehat{S}_{β} of $\mathcal{H}_q(\mathfrak{S}_{(m+1, \dots, n)})$, \overline{z}_{α} and \widehat{z}_{α} are the canonical images of z'_{α} and \widehat{z}_{α} in D_{α} and \widehat{D}_{α} , respectively. Note that by [8, Theorem 3.5] and [32, (5.2), (5.3)], the right ideal of $\mathcal{H}_q(\mathfrak{S}_m)$ generated by z'_{β} is isomorphic to the dual Specht module S_{β} , and the dual Specht module S_{β} is also isomorphic to the right ideal of $\mathcal{H}_q(\mathfrak{S}_m)$ generated by $x_{\beta'}^{\#} T_{w_{\beta'}}^{\#} y_{\beta}^{\#}$. Here “#” denotes the K -algebra automorphism of $\mathcal{H}_q(\mathfrak{S}_n)$ which is defined on generators by $T_i^{\#} := (-q) T_i^{-1}$ for each integer i with $1 \leq i \leq m-1$. We take this chance to point out that the definition of $S(\lambda)_{\pm}$ and $D(\mu)_{\pm}$ given in [19, p. 428, lines –11, –12] are not correct (although this does not affect any other results in [19]). The correct definition should be as what we have given above. By direct verification, we see that

$$F(S_{(\beta,\beta)}^\pm) \cong S(\beta, \beta)_\pm, \quad F(D_{(\alpha,\alpha)}^\pm) \cong D(\alpha, \alpha)_\pm,$$

and for any $\underline{\lambda}, \underline{\mu} \in \mathcal{P}_n$ with $\underline{\mu}$ being e -restricted,

$$F(S_{\underline{\lambda}} \downarrow \mathcal{H}_q(D_n)) \cong S(\underline{\lambda}), \quad F(D_{\underline{\mu}} \downarrow \mathcal{H}_q(D_n)) \cong D(\underline{\mu}).$$

The following lemma is a direct consequence of this Morita equivalence.

Lemma 2.4. *Let $\underline{\lambda}$ be a bipartition of n and $\underline{\mu}$ an e -restricted bipartition of n . Let β be a partition of m and α an e -restricted partition of m . Then we have*

(2.4.1) *if $|\lambda^{(1)}| \neq |\lambda^{(2)}|$ and $\mu^{(1)} \neq \mu^{(2)}$, then*

$$[S_{\underline{\lambda}} \downarrow \mathcal{H}_q(D_n) : D_{\underline{\mu}} \downarrow \mathcal{H}_q(D_n)] = \begin{cases} [S_{\lambda^{(1)}} : D_{\mu^{(1)}}][\widehat{S}_{\lambda^{(2)}} : \widehat{D}_{\mu^{(2)}}], & \text{if } |\lambda^{(1)}| = |\mu^{(1)}| \text{ \& } |\lambda^{(2)}| = |\mu^{(2)}|, \\ 0, & \text{otherwise;} \end{cases}$$

$$[S_{\underline{\lambda}} \downarrow \mathcal{H}_q(D_n) : D_{(\alpha,\alpha)}^+] = 0 = [S_{\underline{\lambda}} \downarrow \mathcal{H}_q(D_n) : D_{(\alpha,\alpha)}^-];$$

(2.4.2) *if $|\lambda^{(1)}| = |\lambda^{(2)}| = m$, $\lambda^{(1)} \neq \lambda^{(2)}$ and $|\mu^{(1)}| \neq |\mu^{(2)}|$, then*

$$[S_{\underline{\lambda}} \downarrow \mathcal{H}_q(D_n) : D_{\underline{\mu}} \downarrow \mathcal{H}_q(D_n)] = 0;$$

$$[S_{(\beta,\beta)}^+ : D_{\underline{\mu}} \downarrow \mathcal{H}_q(D_n)] = 0 = [S_{(\beta,\beta)}^- : D_{\underline{\mu}} \downarrow \mathcal{H}_q(D_n)].$$

Theorem 2.5. *Let $\underline{\lambda}$ be a bipartition of n with $|\lambda^{(1)}| = |\lambda^{(2)}| = m$. Let $\underline{\mu}$ be an e -restricted bipartition of n with $|\mu^{(1)}| = |\mu^{(2)}| = m$. Let β be a partition of m and α an e -restricted partition of m . Then we have*

(2.5.1) *if $\lambda^{(1)} \neq \lambda^{(2)}$ and $\mu^{(1)} \neq \mu^{(2)}$, then*

$$[S_{\underline{\lambda}} \downarrow \mathcal{H}_q(D_n) : D_{\underline{\mu}} \downarrow \mathcal{H}_q(D_n)] = [S_{\lambda^{(1)}} : D_{\mu^{(1)}}][\widehat{S}_{\lambda^{(2)}} : \widehat{D}_{\mu^{(2)}}] + [S_{\lambda^{(1)}} : D_{\mu^{(2)}}][\widehat{S}_{\lambda^{(2)}} : \widehat{D}_{\mu^{(1)}}],$$

$$[S_{\underline{\lambda}} \downarrow \mathcal{H}_q(D_n) : D_{(\alpha,\alpha)}^+] = [S_{\underline{\lambda}} \downarrow \mathcal{H}_q(D_n) : D_{(\alpha,\alpha)}^-] = [S_{\lambda^{(1)}} : D_{\alpha}][\widehat{S}_{\lambda^{(2)}} : \widehat{D}_{\alpha}]/2;$$

(2.5.2) *if $\mu^{(1)} \neq \mu^{(2)}$, then*

$$[S_{(\beta,\beta)}^+ : D_{\underline{\mu}} \downarrow \mathcal{H}_q(D_n)] = [S_{\beta} : D_{\mu^{(1)}}][\widehat{S}_{\beta} : \widehat{D}_{\mu^{(2)}}]/2 + [S_{\beta} : D_{\mu^{(2)}}][\widehat{S}_{\beta} : \widehat{D}_{\mu^{(1)}}]/2,$$

$$[S_{(\beta,\beta)}^- : D_{\underline{\mu}} \downarrow \mathcal{H}_q(D_n)] = [S_{(\beta,\beta)}^+ : D_{\underline{\mu}} \downarrow \mathcal{H}_q(D_n)].$$

Proof. Using the functor F , it is easy to see that the first equality in (2.5.1) follows from the exactness of the induction functor $\uparrow_{\mathcal{H}_q(\mathfrak{S}_{(m,m)})}^{A(m)}$. The other equalities follow from (2.3.4), (2.3.5) and the following Morita equivalence [9]:

$$\mathcal{H}_q(B_n) \stackrel{\text{Morita}}{\sim} \bigoplus_{a=0}^n \mathcal{H}_q(\mathfrak{S}_{(a,n-a)}). \quad \square \quad (2.6)$$

It remains to determine the decomposition numbers

$$[S_{(\beta,\beta)}^+ : D_{(\alpha,\alpha)}^\pm], \quad [S_{(\beta,\beta)}^- : D_{(\alpha,\alpha)}^\pm].$$

Lemma 2.7. Let β be a partition of m and α an e -restricted partition of m . Then we have

$$(2.7.1) \quad [S_{(\beta,\beta)}^+ : D_{(\alpha,\alpha)}^+] = [S_{(\beta,\beta)}^- : D_{(\alpha,\alpha)}^-];$$

$$(2.7.2) \quad [S_{(\beta,\beta)}^+ : D_{(\alpha,\alpha)}^-] = [S_{(\beta,\beta)}^- : D_{(\alpha,\alpha)}^+];$$

$$(2.7.3) \quad [S_{(\beta,\beta)}^+ : D_{(\alpha,\alpha)}^+] + [S_{(\beta,\beta)}^+ : D_{(\alpha,\alpha)}^-] = [S_\beta : D_\alpha]^2.$$

Proof. The first two equalities follow from (2.3.4), while the last equality follows from (2.6) and (2.3.4) and the fact that

$$S_{(\beta,\beta)} \downarrow \mathcal{H}_q(D_n) \cong S_{(\beta,\beta)}^+ \oplus S_{(\beta,\beta)}^-, \quad D_{(\alpha,\alpha)} \downarrow \mathcal{H}_q(D_n) \cong D_{(\alpha,\alpha)}^+ \oplus D_{(\alpha,\alpha)}^-. \quad \square$$

Therefore, it suffices to determine the decomposition number $[S_{(\beta,\beta)}^+ : D_{(\alpha,\alpha)}^+]$ for each partition β of m and each e -restricted partition α of m . Note that these decomposition numbers are the 2-splittable decomposition numbers in the sense of [26].

The purpose of this article is to give an explicit formula for these decomposition numbers. We shall relate these decomposition numbers with the decomposition numbers of the Hecke algebra $\mathcal{H}_q(\mathfrak{S}_m)$ and certain Schur elements of $\mathcal{H}_q(B_n)$ and of $\mathcal{H}_q(\mathfrak{S}_m)$. The main results of this paper are the following two theorems:

Theorem 2.8. Let β be a partition of m and α an e -restricted partition of m . Let

$$d_{\beta,\alpha} := [S_\beta : D_\alpha].$$

Then we have the following equality in K :

$$[S_{(\beta,\beta)}^+ : D_{(\alpha,\alpha)}^+] = d_{\beta,\alpha} \left(\frac{\sqrt{f_{(\beta,\beta)}(q)}}{\sqrt{f_{(\alpha,\alpha)}(q)}} + d_{\beta,\alpha} \right) / 2.$$

In particular, if $\text{char } K = 0$, then the above equality completely determines the decomposition number.

Note that *a priori* we do not even know why the right-hand side term in the above equality should be an element in the prime subfield of K . Note also that we allow $q = 1$ in the above theorem. The Laurent polynomials $f_{(\beta,\beta)}(v)$, $f_{(\alpha,\alpha)}(v)$ appeared in the above theorem can be computed explicitly by the next theorem.

Theorem 2.9. Let $\underline{\lambda} = (\lambda^{(1)}, \lambda^{(2)})$ be an arbitrary bipartition of n . Then we have

$$f_{\underline{\lambda}}(v) = v^{\frac{n(n-1)}{2}} \frac{s_{\underline{\lambda}}(v, 1)}{s_{\lambda^{(1)}}(v) s_{\lambda^{(2)}}(v)},$$

where $s_{\underline{\lambda}}(v, \tilde{v})$ (resp., $s_{\lambda^{(1)}}(v)$, $s_{\lambda^{(2)}}(v)$) is the Schur element corresponding to $\underline{\lambda}$ (resp., corresponding to $\lambda^{(1)}$, $\lambda^{(2)}$), and \tilde{v} is another indeterminate over \mathbb{Z} .

Note that these Schur elements are some explicit defined Laurent polynomials on v , \tilde{v} . For example, if λ is a partition, then

$$s_\lambda(v) = v^{-\ell(w_{\lambda',0})} \prod_{(i,j) \in [\lambda]} [h_{i,j}^\lambda]_v,$$

where $h_{i,j}^\lambda = \lambda_i + \lambda'_j - i - j + 1$ (the (i, j) th hook length), $w_{\lambda',0}$ is the unique longest element in $\mathfrak{S}_{\lambda'}$, λ' denotes the conjugate partition of λ , and for each integer k ,

$$[k]_v := \frac{v^k - 1}{v - 1} \in \mathcal{A}.$$

We refer the reader to [31] for the explicit definitions of Schur elements corresponding to arbitrary multi-partitions.

Example 2.10. Suppose that $K = \mathbb{C}$, $n = 6$, $\beta = (2, 1)$, $\alpha = (1, 1, 1)$. If $e = 3$, then α is e -restricted and the assumption (2.1) is satisfied. In this case, let q be a primitive 3th root of unity in \mathbb{C} . Then $1 + q + q^2 = 0$. It is well known that $[S_\beta : D_\alpha] = 1$. Applying Theorem 2.9 and the known formulae for Schur elements, we get that

$$\begin{aligned} f_{(\beta,\beta)}(v) &= v^4(v+1)^4(v^3+1)^2, \\ f_{(\alpha,\alpha)}(v) &= (v+1)^2(v^2+1)^2(v^3+1)^2. \end{aligned}$$

Applying Theorem 2.8, we get that

$$[S_{(\beta,\beta)}^+ : D_{(\alpha,\alpha)}^+] = \frac{1}{2} \left(\frac{q^2(q+1)^2(q^3+1)}{(q+1)(q^2+1)(q^3+1)} + 1 \right) = 1.$$

If $e = 5$, then α is also e -restricted and the assumption (2.1) is still satisfied. In that case, $[S_\beta : D_\alpha] = 0$, and hence

$$[S_{(\beta,\beta)}^+ : D_{(\alpha,\alpha)}^+] = 0.$$

3. Lifting to q -Schur algebras

For each integer $i \in \{1, 2, \dots, n-1\} \setminus \{m\}$, we define

$$\hat{i} = \begin{cases} i + m, & \text{if } 1 \leq i < m, \\ i - m, & \text{if } m < i \leq n-1. \end{cases}$$

Let $\hat{}$ be the group automorphism of $\mathfrak{S}_{(m,m)} := \mathfrak{S}_m \times \mathfrak{S}_{\{m+1,\dots,n\}}$ which is defined on generators by $\hat{s}_i = s_{\hat{i}}$ for each integer $i \in \{1, 2, \dots, n-1\} \setminus \{m\}$. We use the same notation $\hat{}$ to denote the algebra automorphism of the Hecke algebra $\mathcal{H}_q(\mathfrak{S}_{(m,m)}) = \mathcal{H}_q(\mathfrak{S}_m) \otimes \mathcal{H}_q(\mathfrak{S}_{\{m+1,\dots,n\}})$ which is defined on generators by $\hat{T}_i = T_{\hat{i}}$ for each integer $i \in \{1, 2, \dots, n-1\} \setminus \{m\}$.

For each partition λ of n , we set

$$x_\lambda := \sum_{w \in \mathfrak{S}_\lambda} T_w,$$

where \mathfrak{S}_λ is the Young subgroup of \mathfrak{S}_n corresponding to λ . We use $\Lambda(n)$ to denote the set of compositions $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ of n into n parts (each part λ_i being nonnegative). Let $\Lambda^+(n)$ be the set of partitions in $\Lambda(n)$. Following Dipper and James [10,11], we define the q -Schur algebra $S_q(n)$ to be

$$S_q(n) := \text{End}_{\mathcal{H}_q(\mathfrak{S}_n)} \left(\bigoplus_{\lambda \in \Lambda(n)} x_\lambda \mathcal{H}_q(\mathfrak{S}_n) \right).$$

In a similar way, we define the q -Schur algebras $S_q(m), \widehat{S}_q(m)$ by using the Hecke algebras $\mathcal{H}_q(\mathfrak{S}_m), \mathcal{H}_q(\mathfrak{S}_{\{m+1, \dots, n\}})$, respectively. Note that to define the q -Schur algebra $\widehat{S}_q(m)$, one needs to use the element \widehat{x}_λ for $\lambda \in \Lambda(m)$.

For any $\lambda, \mu \in \Lambda(n)$, let $\mathfrak{D}_{\lambda, \mu}$ be the set of distinguished \mathfrak{S}_λ - \mathfrak{S}_μ -double coset representatives in \mathfrak{S}_n . Following [11], for each $d \in \mathfrak{D}_{\lambda, \mu}$, we define

$$\phi_{\lambda, \mu}^d \in \text{Hom}_{\mathcal{H}_q(\mathfrak{S}_n)}(x_\mu \mathcal{H}_q(\mathfrak{S}_n), x_\lambda \mathcal{H}_q(\mathfrak{S}_n))$$

by

$$\phi_{\lambda, \mu}^d(x_\mu h) = \sum_{w \in \mathfrak{S}_\lambda d \mathfrak{S}_\mu} T_w h, \quad \forall h \in \mathcal{H}_q(\mathfrak{S}_n).$$

By [11, Theorem 1.4], the elements in the set

$$\{\phi_{\lambda, \mu}^d \mid \lambda, \mu \in \Lambda(n), d \in \mathfrak{D}_{\lambda, \mu}\}$$

form a K -basis of $S_q(n)$ which shall be called *standard bases* in this paper.

For any $\lambda, \mu \in \Lambda(n)$, by [11, (2.3)],

$$\phi_{\lambda, \lambda}^1 S_q(n) \phi_{\mu, \mu}^1 \cong \text{Hom}_{\mathcal{H}_q(\mathfrak{S}_n)}(x_\mu \mathcal{H}_q(\mathfrak{S}_n), x_\lambda \mathcal{H}_q(\mathfrak{S}_n)).$$

Let ω_n denote the partition $(1^n) = \underbrace{(1, 1, \dots, 1)}_{n \text{ copies}}$ of n . Using the natural isomorphism

$$\text{Hom}_{\mathcal{H}_q(\mathfrak{S}_n)}(\mathcal{H}_q(\mathfrak{S}_n), \mathcal{H}_q(\mathfrak{S}_n)) \cong \mathcal{H}_q(\mathfrak{S}_n),$$

we can identify $\mathcal{H}_q(\mathfrak{S}_n)$ with the non-unital K -subalgebra $\phi_{\omega_n, \omega_n}^1 S_q(n) \phi_{\omega_n, \omega_n}^1$ of $S_q(n)$. We use ι_n to denote the resulting injection from $\mathcal{H}_q(\mathfrak{S}_n)$ into $S_q(n)$. In a similar way, we can identify $\mathcal{H}_q(\mathfrak{S}_m)$ with the non-unital K -subalgebra $\phi_{\omega_m, \omega_m}^1 S_q(m) \phi_{\omega_m, \omega_m}^1$ via an injective map ι_m , and $\mathcal{H}_q(\mathfrak{S}_{\{m+1, \dots, n\}})$ with the non-unital K -subalgebra $\widehat{\phi}_{\omega_m, \omega_m}^1 \widehat{S}_q(m) \widehat{\phi}_{\omega_m, \omega_m}^1$ via an injective map $\widehat{\iota}_m$. Note that here in order not to confuse with the standard basis elements of $S_q(m)$, we denote the standard basis element of $\widehat{S}_q(m)$ by $\widehat{\phi}_{\lambda, \mu}^d$.

Let ρ denote the natural injective map from $\mathcal{H}_q(\mathfrak{S}_m) \otimes \mathcal{H}_q(\mathfrak{S}_{\{m+1, \dots, n\}})$ into $\mathcal{H}_q(\mathfrak{S}_n)$. We are going to lift this map to an injection $\widetilde{\rho}$ from $S_q(m) \otimes \widehat{S}_q(m)$ into $S_q(n)$. Let $\mathfrak{D}_{(m, m)}$ be the set of distinguished right coset representatives of $\mathfrak{S}_{(m, m)}$ in \mathfrak{S}_n . Any element $h \in \mathcal{H}_q(\mathfrak{S}_n)$ can be written uniquely as

$$h = \sum_{\substack{w_1 \in \mathfrak{S}_m, w_2 \in \mathfrak{S}_{\{m+1, \dots, n\}} \\ d \in \mathfrak{D}_{(m, m)}}} A_{w_1, w_2}^d T_{w_1} T_{w_2} T_d, \quad (3.1)$$

where $A_{w_1, w_2}^d \in K$ for each w_1, w_2, d . For any $f \in S_q(m)$, $g \in \widehat{S}_q(m)$, we define $\widetilde{\rho}(f \otimes g) \in S_q(n)$ as follows: for any $\mu \in \Lambda(n)$, and any $h \in \mathcal{H}_q(\mathfrak{S}_n)$ which is given by (3.1), if $\mu = (\mu^{(1)}, \mu^{(2)})$, where $\mu^{(1)} \in \Lambda(m)$, then we set

$$\widetilde{\rho}(f \otimes g)(x_\mu h) := \sum_{\substack{w_1 \in \mathfrak{S}_m, w_2 \in \mathfrak{S}_{\{m+1, \dots, n\}} \\ d \in \mathfrak{D}_{(m, m)}}} A_{w_1, w_2}^d f(x_{\mu^{(1)}} T_{w_1}) g(x_{\widehat{\mu}^{(2)}} T_{w_2}) T_d;$$

otherwise, we set $\widetilde{\rho}(f \otimes g)(x_\mu h) := 0$.

Lemma 3.2. *With the notations as above, we have that*

$$\tilde{\rho}(f \otimes g) \in S_q(n).$$

Proof. This is clear by using the fact that

$$(x_{\mu(1)} \mathcal{H}_q(\mathfrak{S}_m) \otimes \hat{x}_{\mu(2)} \mathcal{H}_q(\mathfrak{S}_{\{m+1, \dots, n\}})) \uparrow^{\mathcal{H}_q(\mathfrak{S}_n)}_{\mathcal{H}_q(\mathfrak{S}_{(m,m)})} \cong x_{\mu} \mathcal{H}_q(\mathfrak{S}_n). \quad \square$$

Lemma 3.3. *We have the following commutative diagram of algebra maps:*

$$\begin{array}{ccc} \mathcal{H}_q(\mathfrak{S}_m) \otimes \mathcal{H}_q(\mathfrak{S}_{\{m+1, \dots, n\}}) & \xrightarrow{\rho} & \mathcal{H}_q(\mathfrak{S}_n) \\ \downarrow \iota_m \otimes \hat{\iota}_m & & \downarrow \iota_n \\ S_q(m) \otimes \hat{S}_q(m) & \xrightarrow{\tilde{\rho}} & S_q(n) \end{array}$$

Moreover, $\tilde{\rho}(\phi_{\omega_m, \omega_m}^1 \otimes \hat{\phi}_{\omega_m, \omega_m}^1) = \phi_{\omega_n, \omega_n}^1$ and $\tilde{\rho}$ is an injection.

Proof. This follows from direct verification. \square

Note that the unit element of $S_q(m) \otimes \hat{S}_q(m)$, i.e.,

$$\left(\sum_{\lambda \in \Lambda(m)} \phi_{\lambda, \lambda}^1 \right) \otimes \left(\sum_{\lambda \in \Lambda(m)} \hat{\phi}_{\lambda, \lambda}^1 \right)$$

is not mapped by $\tilde{\rho}$ to the unit element of $S_q(n)$. Henceforth, we identify $S_q(m) \otimes \hat{S}_q(m)$ with a non-unital K -subalgebra of $S_q(n)$ via the injection $\tilde{\rho}$. Recall that ([19, Remark 2.4]) the algebra $A(m)$ was generated by $T_1, \dots, T_{m-1}, T_{m+1}, \dots, T_{n-1}, h(m)$ and satisfy the following relations

$$(T_i + 1)(T_i - q) = 0, \quad \text{for } 1 \leq i \leq n-1, i \neq m, \quad h(m)^2 = z_{m,m},$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad \text{for } 1 \leq i \leq n-2, i \notin \{m-1, m\}$$

$$T_i T_j = T_j T_i, \quad \text{for } 1 \leq i < j-1 \leq n-2, i, j \neq m$$

$$T_i h(m) = \begin{cases} h(m) T_{i+m} & \text{if } 1 \leq i < m, \\ h(m) T_{i-m} & \text{if } m < i \leq n-1, \end{cases}$$

where $z_{m,m}$ is a central element in the Hecke algebra $\mathcal{H}_q(\mathfrak{S}_{(m,m)})$ (see [9]). We are going to lift the elements $h(m), z_{m,m}$ to the corresponding q -Schur algebras.

Lemma 3.4. *Let z be an element in the center of $\mathcal{H}_q(\mathfrak{S}_n)$. We define $Z \in S_q(n)$ by*

$$Z(x_{\lambda} h) = x_{\lambda} h z, \quad \forall \lambda \in \Lambda(n), h \in \mathcal{H}_q(\mathfrak{S}_n).$$

Then Z lies in the center of $S_q(n)$.

Proof. The condition that z is central in $\mathcal{H}_q(\mathfrak{S}_n)$ implies that the above-defined map Z is indeed a right $\mathcal{H}_q(\mathfrak{S}_n)$ -homomorphism. Hence $Z \in S_q(n)$. It remains to check that $\phi_{\lambda,\mu}^d Z = Z \phi_{\lambda,\mu}^d$ for any $\lambda, \mu \in \Lambda(n)$ and any $d \in \mathcal{D}_{\lambda,\mu}$. Applying [11, (2.1)], and by definition, for any $h \in \mathcal{H}_q(\mathfrak{S}_n)$,

$$\begin{aligned}\phi_{\lambda,\mu}^d Z(x_\mu h) &= (\phi_{\lambda,\mu}^d)(x_\mu h z) = \sum_{w \in \mathfrak{S}_\lambda d \mathfrak{S}_\mu} T_w h z, \\ Z \phi_{\lambda,\mu}^d(x_\mu h) &= Z \sum_{w \in \mathfrak{S}_\lambda d \mathfrak{S}_\mu} T_w h = \sum_{w \in \mathfrak{S}_\lambda d \mathfrak{S}_\mu} T_w h z.\end{aligned}$$

This proves that $\phi_{\lambda,\mu}^d Z = Z \phi_{\lambda,\mu}^d$, as required. \square

Definition 3.5. We define $H(m), Z_{m,m} \in S_q(n)$ as follows: for any $\lambda \in \Lambda(n), h \in \mathcal{H}_q(\mathfrak{S}_n)$,

$$\begin{aligned}H(m)(x_\lambda h) &:= \begin{cases} x_\lambda h(m)h, & \text{if } \lambda = (\lambda^{(1)}, \lambda^{(2)}) \text{ with } \lambda^{(1)} \in \Lambda(m), \\ 0, & \text{otherwise,} \end{cases} \\ Z_{m,m}(x_\lambda h) &:= \begin{cases} x_\lambda Z_{m,m}h, & \text{if } \lambda = (\lambda^{(1)}, \lambda^{(2)}) \text{ with } \lambda^{(1)} \in \Lambda(m), \\ 0, & \text{otherwise.} \end{cases}\end{aligned}$$

Lemma 3.6. With the notations as above, we have that

$$H(m)^2 = Z_{m,m},$$

and $Z_{m,m}$ lies in the center of $S_q(m) \otimes \widehat{S}_q(m)$. Moreover $Z_{m,m}$ is invertible in $S_q(m) \otimes \widehat{S}_q(m)$.

Proof. The equality $H(m)^2 = Z_{m,m}$ follows from direct verification and the fact that $h(m)^2 = z_{m,m}$.

Since $z_{m,m}$ lies in $\mathcal{H}_q(\mathfrak{S}_{(m,m)})$ and $z_{m,m}$ is invertible in $\mathcal{H}_q(\mathfrak{S}_{(m,m)})$ (see [9, (4.12)]), it follows that $Z_{m,m} \in S_q(m) \otimes \widehat{S}_q(m)$ and $Z_{m,m}$ is invertible in $S_q(m) \otimes \widehat{S}_q(m)$. The inverse of $Z_{m,m}$ is given by

$$Z_{m,m}^{-1}(x_\lambda h) := \begin{cases} x_\lambda Z_{m,m}^{-1}h, & \text{if } \lambda = (\lambda^{(1)}, \lambda^{(2)}) \text{ with } \lambda^{(1)} \in \Lambda(m), \\ 0, & \text{otherwise,} \end{cases}$$

for any $\lambda \in \Lambda(n), h \in \mathcal{H}_q(\mathfrak{S}_n)$. The claim that $Z_{m,m}$ lies in the center of $S_q(m) \otimes \widehat{S}_q(m)$ also follows from direct verification. \square

Note that $Z_{m,m}$ is invertible in $S_q(m) \otimes \widehat{S}_q(m)$ does not mean that $Z_{m,m}$ is invertible in $S_q(n)$. The point is that the unit element of $S_q(n)$ is not the same as the unit element of $S_q(m) \otimes \widehat{S}_q(m)$. By the same reason, we know that $H(m)$ is not an invertible element in $S_q(n)$.

We identify $\mathcal{H}_q(\mathfrak{S}_{(m,m)})$ with a subalgebra of $S_q(m) \otimes \widehat{S}_q(m)$ via the injection $\iota_m \otimes \widehat{\iota}_m$. For any $\lambda, \mu \in \Lambda(m)$, let $\mathcal{D}_{\lambda,\mu}$ be the set of distinguished \mathfrak{S}_λ - \mathfrak{S}_μ -double coset representatives in \mathfrak{S}_m . For any $d \in \mathcal{D}_{\lambda,\mu}$, it is easy to see that \widehat{d} is a distinguished \mathfrak{S}_λ - \mathfrak{S}_μ -double coset representative in $\mathfrak{S}_{\{m+1,\dots,n\}}$. Here $\widehat{\mathfrak{S}}_\lambda, \widehat{\mathfrak{S}}_\mu$ denote the image of $\mathfrak{S}_\lambda, \mathfrak{S}_\mu$ under the automorphism $\widehat{}$.

Lemma 3.7. The automorphism $\widehat{}$ of $\mathcal{H}_q(\mathfrak{S}_{(m,m)})$ can be uniquely extended to a K -algebra automorphism (still denoted by $\widehat{}$) of $S_q(m) \otimes \widehat{S}_q(m)$ such that for any $\lambda, \mu \in \Lambda(m)$ and any $d \in \mathcal{D}_{\lambda,\mu}$,

$$\widehat{\phi_{\lambda,\mu}^d} = \widehat{\phi}_{\lambda,\mu}^{\widehat{d}}.$$

Furthermore, $H(m)\phi_{\lambda,\mu}^d = \widehat{\phi}_{\lambda,\mu}^{\widehat{d}}H(m)$ and $\phi_{\lambda,\mu}^d H(m) = H(m)\widehat{\phi}_{\lambda,\mu}^{\widehat{d}}$.

Proof. For the first claim, it suffices to show that the map which sends $\phi_{\lambda,\mu}^d$ to $\widehat{\phi}_{\lambda,\mu}^d$, for any $\lambda, \mu \in \Lambda(m)$ and any $d \in \mathcal{D}_{\lambda,\mu}$, can be uniquely extends to a K -algebra map. To this end, it is enough to show that for any $\lambda, \mu, \nu \in \Lambda(m)$ and any $d \in \mathcal{D}_{\lambda,\mu}, d' \in \mathcal{D}_{\mu,\nu}$,

$$\widehat{\phi}_{\lambda,\mu}^d \widehat{\phi}_{\mu,\nu}^{d'} = \widehat{\phi_{\lambda,\mu}^d \phi_{\mu,\nu}^{d'}}. \quad (3.8)$$

Suppose that $\phi_{\lambda,\mu}^d \phi_{\mu,\nu}^{d'} = \sum_{d'' \in \mathcal{D}_{\lambda,\nu}} A_{d''} \phi_{\lambda,\nu}^{d''}$, where $A_{d''} \in K$ for each d'' . Then

$$\widehat{\phi_{\lambda,\mu}^d \phi_{\mu,\nu}^{d'}} = \sum_{d'' \in \mathcal{D}_{\lambda,\nu}} A_{d''} \widehat{\phi_{\lambda,\nu}^{d''}}.$$

By definition, it is easy to verify that for any $h \in \mathcal{H}_q(\mathfrak{S}_m)$,

$$\widehat{\phi_{\mu,\nu}^{d'}}(\widehat{x}_\nu \widehat{h}) = \widehat{\phi_{\mu,\nu}^{d'}}(x_\nu h), \quad \widehat{\phi_{\lambda,\mu}^d}(\widehat{x}_\mu \widehat{h}) = \widehat{\phi_{\lambda,\mu}^d}(x_\mu h).$$

Therefore,

$$\widehat{\phi_{\lambda,\mu}^d \phi_{\mu,\nu}^{d'}}(\widehat{x}_\nu) = \widehat{\phi_{\lambda,\mu}^d \phi_{\mu,\nu}^{d'}}(x_\nu) = \sum_{d'' \in \mathcal{D}_{\lambda,\nu}} A_{d''} \widehat{\phi_{\lambda,\nu}^{d''}}(x_\nu) = \widehat{\phi_{\lambda,\mu}^d \phi_{\mu,\nu}^{d'}}(\widehat{x}_\nu).$$

This proves (3.8), as required. The proof of the last two equalities are straightforward and will be omitted. \square

Definition 3.9. Let $\widetilde{A}(m)$ denote the non-unital K -subalgebra of $S_q(n)$ generated by $S_q(m) \otimes \widehat{S}_q(m)$ and $H(m)$.

The algebra $\widetilde{A}(m)$ can be regarded as a covering of the algebra $A(m)$. Note that the unit element of $\widetilde{A}(m)$ is the unit element of $S_q(m) \otimes \widehat{S}_q(m)$, which is different from the unit element of $S_q(n)$. Note also that, by Lemma 3.6, although $H(m)$ is not invertible in $S_q(n)$, $H(m)$ is indeed an invertible element in the algebra $\widetilde{A}(m)$.

Definition 3.10. For each partition $\lambda \in \Lambda^+(m)$, we define

$$Z_\lambda := y_{\lambda'} T_{w_{\lambda'}} \phi_{\lambda, \omega_m}^1.$$

By [11], the right ideal of $S_q(m)$ generated by Z_λ is the Weyl module Δ_λ of $S_q(m)$ with highest weight λ . In a similar way, we can define the element \widehat{Z}_λ and the Weyl module $\widehat{\Delta}_\lambda$ of $\widehat{S}_q(m)$.

Lemma 3.11. Let $\lambda, \mu \in \Lambda^+(m)$. Then we have

$$(z'_\lambda \otimes \widehat{z}'_\mu) z_{m,m} = f_{(\lambda,\mu)}(q) (z'_\lambda \otimes \widehat{z}'_\mu), \quad (Z_\lambda \otimes \widehat{Z}_\mu) z_{m,m} = f_{(\lambda,\mu)}(q) (Z_\lambda \otimes \widehat{Z}_\mu).$$

Proof. Since the dual Specht module S_λ is also isomorphic to the right ideal of $\mathcal{H}_q(\mathfrak{S}_m)$ generated by $x_\lambda^\# T_{w_{\lambda'}}^\# y_\lambda^\#$, the first equality follows from [19, Lemma 3.2]. The second equality follows from the first equality and the definitions of $z_{m,m}, Z_\lambda$ and \widehat{Z}_λ . \square

To simplify notations, we shall denote $H(m)$ by θ and $S_q(m) \otimes \widehat{S}_q(m)$ by $S_q^{m,m}$. The set of subspaces $\{S_q^{m,m}, \theta S_q^{m,m}\}$ is a $\mathbb{Z}/2\mathbb{Z}$ -Clifford system in $\widetilde{A}(m)$ in the sense of [6, (11.12)]. For any $\lambda, \mu \in \Lambda^+(m)$, we set

$$\Delta_{\lambda,\mu} := \Delta_\lambda \otimes \widehat{\Delta}_\mu, \quad L_{\lambda,\mu} := L_\lambda \otimes \widehat{L}_\mu,$$

where $\Delta_\lambda, L_\lambda$ (resp., $\widehat{\Delta}_\mu, \widehat{L}_\mu$) are the Weyl module, irreducible module of the q -Schur algebra $S_q(m)$ (resp., $\widehat{S}_q(m)$), respectively. By definition, L_λ (resp., \widehat{L}_μ) is the unique simple $S_q(m)$ -head (resp., $\widehat{S}_q(m)$ -head) of Δ_λ (resp., of $\widehat{\Delta}_\mu$). Hence $L_{\lambda,\mu}$ is the unique simple $S_q^{m,m}$ -head of $\Delta_{\lambda,\mu}$.

Note that by Lemma 3.7, the automorphism $\widehat{\cdot}$ of $S_q^{m,m}$ is induced by θ . For any $S_q^{m,m}$ -module M , let M^θ be the new $S_q^{m,m}$ -module obtained by twisting the action of $S_q^{m,m}$ by $\widehat{\cdot}$. We have the following result.

Lemma 3.12. *For any $\lambda, \mu \in \Lambda^+(m)$, we have that*

$$(\Delta_{\lambda,\mu})^\theta \cong \Delta_{\mu,\lambda}, \quad (L_{\lambda,\mu})^\theta \cong L_{\mu,\lambda}.$$

Proof. This follows directly from Lemma 3.7. \square

Definition 3.13. For any $\lambda, \mu \in \Lambda^+(m)$, we set

$$\widetilde{\Delta}_{\lambda,\mu} := \Delta_{\lambda,\mu} \uparrow_{S_q^{m,m}}^{\widetilde{A}(m)}, \quad \widetilde{L}_{\lambda,\mu} := L_{\lambda,\mu} \uparrow_{S_q^{m,m}}^{\widetilde{A}(m)}.$$

Let $\widetilde{\sigma}$ be the automorphism of $\widetilde{A}(m)$ which is defined on generators by

$$\theta^j x \mapsto (-1)^j \theta^j x, \quad \forall x \in S_q^{m,m}, \quad j \in \mathbb{Z}.$$

Clearly, $\widetilde{\sigma} \downarrow_{S_q^{m,m}} = \text{Id}$. By Lemma 3.6, we can apply [17, (2.2)] and [24, Appendix]. That is, as $\widetilde{A}(m)$ - $\widetilde{A}(m)$ -bimodule,

$$\widetilde{A}(m) \otimes_{S_q^{m,m}} \widetilde{A}(m) \cong \widetilde{A}(m) \oplus (\widetilde{A}(m))^{\widetilde{\sigma}}, \quad (3.14)$$

where the left $\widetilde{A}(m)$ -module structure on $(\widetilde{A}(m))^{\widetilde{\sigma}}$ was just given by left multiplication, while the right $\widetilde{A}(m)$ -module structure on $(\widetilde{A}(m))^{\widetilde{\sigma}}$ was given by right multiplication twisted by $\widetilde{\sigma}$.

Lemma 3.15. *Let $\lambda, \mu \in \Lambda^+(m)$. We have that*

- (1) if $\lambda \neq \mu$, then $\widetilde{L}_{\lambda,\mu} \cong \widetilde{L}_{\mu,\lambda}$ is a simple $\widetilde{A}(m)$ -module;
- (2) there is a direct sum decomposition of $\widetilde{A}(m)$ -module: $\widetilde{L}_{\lambda,\lambda} = \widetilde{L}_{\lambda,\lambda}^+ \oplus \widetilde{L}_{\lambda,\lambda}^-$, where

$$\begin{aligned} \widetilde{L}_{\lambda,\lambda}^+ &:= (\sqrt{f_{(\lambda,\lambda)}(q)} \overline{Z}_\lambda \otimes \widehat{Z}_\lambda + \overline{Z}_\lambda \otimes \widehat{Z}_\lambda \theta) S_q^{m,m}, \\ \widetilde{L}_{\lambda,\lambda}^- &:= (\sqrt{f_{(\lambda,\lambda)}(q)} \overline{Z}_\lambda \otimes \widehat{Z}_\lambda - \overline{Z}_\lambda \otimes \widehat{Z}_\lambda \theta) S_q^{m,m}, \end{aligned}$$

- where \overline{Z}_λ (resp., \widehat{Z}_λ) is the natural image of Z_λ (resp., of \widehat{Z}_λ) in L_λ (resp., \widehat{L}_λ);
- (3) $\widetilde{A}(m)$ is split over K , and the set

$$\{\widetilde{L}_{\lambda,\mu}, \widetilde{L}_{\beta,\beta}^+, \widetilde{L}_{\beta,\beta}^- \mid \lambda, \mu, \beta \vdash m, \lambda \neq \mu, (\lambda, \mu) \in \mathcal{P}_n / \sim\}$$

forms a complete set of pairwise non-isomorphic absolutely simple $\widetilde{A}(m)$ -modules;

- (4) $(\widetilde{L}_{\lambda,\mu})^{\widetilde{\sigma}} \cong \widetilde{L}_{\lambda,\mu} \cong \widetilde{L}_{\mu,\lambda}$, $(\widetilde{L}_{\beta,\beta}^+)^{\widetilde{\sigma}} \cong \widetilde{L}_{\beta,\beta}^-$.

Proof. We only give the proof of (4), as the other claims follow from (4), (3.14) and Frobenius reciprocity.

In fact, one can check that the following map gives the isomorphism $(\tilde{L}_{\lambda,\mu})^{\tilde{\sigma}} \cong \tilde{L}_{\lambda,\mu}$: for any $h_1, h_2 \in S_q^{m,m}$,

$$(\bar{Z}_\lambda \otimes \hat{\bar{Z}}_\mu)h_1 + (\bar{Z}_\lambda \otimes \hat{\bar{Z}}_\mu)h_2\theta \mapsto -f_{(\lambda,\mu)}(q)(\bar{Z}_\lambda \otimes \hat{\bar{Z}}_\mu)h_2 + (\bar{Z}_\lambda \otimes \hat{\bar{Z}}_\mu)h_1\theta;$$

while the following map gives the isomorphism $(\tilde{L}_{\lambda,\lambda}^+)^{\tilde{\sigma}} \cong \tilde{L}_{\lambda,\lambda}^-$: for any $h \in S_q^{m,m}$,

$$(\sqrt{f_{(\lambda,\lambda)}(q)}\bar{Z}_\lambda \otimes \hat{\bar{Z}}_\lambda + \bar{Z}_\lambda \otimes \hat{\bar{Z}}_\lambda\theta)h \mapsto (\sqrt{f_{(\lambda,\lambda)}(q)}\bar{Z}_\lambda \otimes \hat{\bar{Z}}_\lambda - \bar{Z}_\lambda \otimes \hat{\bar{Z}}_\lambda\theta)h. \quad \square$$

There is also a direct sum decomposition of $\tilde{A}(m)$ -module: $\tilde{\Delta}_{\lambda,\lambda} = \tilde{\Delta}_{\lambda,\lambda}^+ \oplus \tilde{\Delta}_{\lambda,\lambda}^-$, where

$$\begin{aligned}\tilde{\Delta}_{\lambda,\lambda}^+ &:= (\sqrt{f_{(\lambda,\lambda)}(q)}Z_\lambda \otimes \hat{Z}_\lambda + Z_\lambda \otimes \hat{Z}_\lambda\theta)S_q^{m,m}, \\ \tilde{\Delta}_{\lambda,\lambda}^- &:= (\sqrt{f_{(\lambda,\lambda)}(q)}Z_\lambda \otimes \hat{Z}_\lambda - Z_\lambda \otimes \hat{Z}_\lambda\theta)S_q^{m,m}.\end{aligned}$$

Lemma 3.16. *With the same notations as above, we have*

- (1) if $\lambda \neq \mu$, then $\tilde{L}_{\lambda,\mu}$ is the unique simple $\tilde{A}(m)$ -head of $\tilde{\Delta}_{\lambda,\mu}$;
- (2) $\tilde{L}_{\lambda,\lambda}^+$ (resp., $\tilde{L}_{\lambda,\lambda}^-$) is the unique simple $\tilde{A}(m)$ -head of $\tilde{\Delta}_{\lambda,\lambda}^+$ (resp., of $\tilde{\Delta}_{\lambda,\lambda}^-$);
- (3) $(\tilde{\Delta}_{\lambda,\mu})^{\tilde{\sigma}} \cong \tilde{\Delta}_{\lambda,\mu} \cong \tilde{\Delta}_{\mu,\lambda}$, $(\tilde{\Delta}_{\lambda,\lambda}^+)^{\tilde{\sigma}} \cong \tilde{\Delta}_{\lambda,\lambda}^-$.

Proof. These follow from Frobenius reciprocity. \square

Corollary 3.17. *Let λ be a partition in $\Lambda^+(m)$. Let θ acts as the scalar $\sqrt{f_{(\lambda,\lambda)}(q)}$ (resp., $-\sqrt{f_{(\lambda,\lambda)}(q)}$) on the highest weight vector of $\Delta_{\lambda,\lambda}$. Then this action can be uniquely extends to a representation of $\tilde{A}(m)$ on $\Delta_{\lambda,\lambda}$. The resulting $\tilde{A}(m)$ -module is isomorphic to $\Delta_{\lambda,\lambda}^+$ (resp., $\Delta_{\lambda,\lambda}^-$). The same statements hold for $L_{\lambda,\lambda}^+$ and $L_{\lambda,\lambda}^-$.*

Note that the set $\{\phi_{\lambda,\lambda}^1\}_{\lambda \in \Lambda(m)}$ is a set of pairwise orthogonal idempotents in $S_q(m)$, and $\sum_{\lambda \in \Lambda(m)} \phi_{\lambda,\lambda}^1 = 1$. For any right $S_q(m)$ -module M , it is clear that

$$M = \bigoplus_{\lambda \in \Lambda(m)} M\phi_{\lambda,\lambda}^1.$$

For each $\lambda \in \Lambda(m)$, we define $M_\lambda := M\phi_{\lambda,\lambda}^1$, and we call M_λ the λ -weight space of $S_q(m)$ -module M . Note also that $S_q(m)$ is an epimorphic image of the quantum algebra $U_K(\mathfrak{gl}_m)$ associated to \mathfrak{gl}_m (cf. [5,15]). Any $S_q(m)$ -module M naturally becomes a module over $U_K(\mathfrak{gl}_m)$. The definition of weight space we used here coincides with the usual definition of weight space for $U_K(\mathfrak{gl}_m)$ -module. In a similar way, we can define the weight space for any $\hat{S}_q(m)$ -module. Therefore, we have also the notion of weight space for any $S_q^{m,m}$ -module. The weights of any $S_q^{m,m}$ -module are elements in the set $\underline{\Lambda}(m) := \Lambda(m) \times \Lambda(m)$.

There is a natural additive group structure on $\underline{\Lambda}(m)$. Let $\{e^\lambda\}_{\lambda \in \underline{\Lambda}(m)}$ denote the standard basis of the group ring $\mathbb{Z}[\underline{\Lambda}(m)]$ over \mathbb{Z} . Then $e^\lambda e^\mu = e^{\lambda+\mu}$. For any finite dimensional $S_q^{m,m}$ -module M , we define the formal character of M as

$$\text{ch } M = \sum_{\lambda \in \underline{\Lambda}(m)} \dim M_\lambda e^\lambda \in \mathbb{Z}[\underline{\Lambda}(m)].$$

For any short exact sequence $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ of finite dimensional $S_q^{m,m}$ -modules, it is clear that

$$\text{ch } M = \text{ch } M' + \text{ch } M''.$$

Therefore, the map ch is a map defined on the Grothendieck group $\mathcal{R}(S_q^{m,m})$ associated to the category of finite dimensional $S_q^{m,m}$ -modules.

Set $e = \phi_{\omega_m, \omega_m}^1 \otimes \hat{\phi}_{\omega_m, \omega_m}^1$. Then e is an idempotent in $\tilde{A}(m)$, and $e\tilde{A}(m)e = A(m)$, $eS_q^{m,m}e = \mathcal{H}_q(\mathfrak{S}_{(m,m)})$. We define a functor \tilde{F} from the category of finite dimensional $\tilde{A}(m)$ -modules to the category of finite dimensional $A(m)$ -modules as follows: for any finite dimensional $\tilde{A}(m)$ -module M, N , and any $\varphi \in \text{Hom}_{\tilde{A}(m)}(M, N)$, $\tilde{F}(M) = Me$, $\tilde{F}(N) = Ne$, and

$$\tilde{F}(\varphi)(xe) := \varphi(x)e, \quad \forall x \in M.$$

Let F be the Schur functor (induced by e) from the category of finite dimensional $S_q^{m,m}$ -modules to the category of finite dimensional $\mathcal{H}_q(\mathfrak{S}_{(m,m)})$ -modules. Then we have the following commutative diagram of functors:

$$\begin{array}{ccc} \text{Mod-}\tilde{A}(m) & \xrightarrow{\text{Res}} & \text{Mod-}S_q^{m,m} \\ \tilde{F} \downarrow & & \downarrow F \\ \text{Mod-}A(m) & \xrightarrow{\text{Res}} & \text{Mod-}\mathcal{H}_q(\mathfrak{S}_{(m,m)}) \end{array} \quad (3.18)$$

We set $\underline{\Delta}^+(m) := \Lambda^+(m) \times \Lambda^+(m)$.

Lemma 3.19. Let $(\lambda, \mu) \in \underline{\Delta}^+(m)$ such that $\lambda \neq \mu$. Then we have

$$\begin{aligned} \tilde{F}(\tilde{\Delta}_{\lambda, \lambda}^{\pm}) &= S(\lambda, \lambda)_{\pm}, & \tilde{F}(\tilde{\Delta}_{\lambda, \mu}) &= S(\lambda, \mu), \\ \tilde{F}(\tilde{L}_{\lambda, \lambda}^{\pm}) &= \begin{cases} D(\lambda, \lambda)_{\pm}, & \text{if } \lambda \text{ is } e\text{-restricted;} \\ 0, & \text{otherwise,} \end{cases} \\ \tilde{F}(\tilde{L}_{\lambda, \mu}) &= \begin{cases} D(\lambda, \mu), & \text{if } (\lambda, \mu) \text{ is } e\text{-restricted;} \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Proof. These follow from (3.18) and direct verification. \square

Lemma 3.20. \tilde{F} is an exact functor. In particular, \tilde{F} induces a homomorphism from the Grothendieck group $\mathcal{R}(\tilde{A}(m))$ associated to the category of finite dimensional $\tilde{A}(m)$ -modules to the Grothendieck group $\mathcal{R}(A(m))$ associated to the category of finite dimensional $A(m)$ -modules.

Proof. This follows from the same arguments as in [18, Section 6]. \square

Corollary 3.21. For any $\lambda, \mu \in \Lambda^+(m)$ with μ being e -restricted, we have the following equality of decomposition numbers:

$$[\tilde{\Delta}_{\lambda, \lambda}^+ : \tilde{L}_{\mu, \mu}^+] = [S(\lambda, \lambda)_+ : D(\mu, \mu)_+] = [S_{(\lambda, \lambda)}^+ : D_{(\mu, \mu)}^+].$$

Proof. This follows directly from Lemmas 3.19 and 3.20. \square

Therefore, computing the decomposition number $[S_{(\lambda, \lambda)}^+ : D_{(\mu, \mu)}^+]$ can be reduced to computing the decomposition number $[\tilde{\Delta}_{\lambda, \lambda}^+ : \tilde{L}_{\mu, \mu}^+]$. The latter will be done in the final section.

4. Computing the Laurent polynomial $f_{\underline{\lambda}}(v)$

Let a be a fixed integer with $0 \leq a \leq n$ and $\underline{\lambda} = (\lambda^{(1)}, \lambda^{(2)})$ be a fixed a -bipartition of n . The purpose of this section is to give a close formula for the Laurent polynomial $f_{\underline{\lambda}}(v)$, which was introduced in [19, Lemma 3.2].

Recall that by [19, Lemma 3.2], certain central element (denoted by $z_{a, n-a}$ in [19, Lemma 3.2]) of $\mathcal{H}_v(\mathfrak{S}_{(a, n-a)})$ acts on the dual Specht module $S_{\underline{\lambda}}$ as the scalar $f_{\underline{\lambda}}(v)$. These central elements $z_{a, n-a}$ arise in the study of certain homomorphism between some (dual) Specht modules over the Hecke algebra $\mathcal{H}_v(B_n)$. Indeed, similar central elements do arise if we consider the more general type B_n Hecke algebra $\mathcal{H}_{v, \tilde{v}}(B_n)$ which has two parameters v, \tilde{v} , where \tilde{v} is another indeterminate over \mathbb{Z} . By some abuse of notations, we will denote the resulting Laurent polynomial by $f_{\underline{\lambda}}(v, \tilde{v})$. The relation with our previous introduced Laurent polynomial $f_{\underline{\lambda}}(v)$ is given by

$$f_{\underline{\lambda}}(v) = f_{\underline{\lambda}}(v, 1).$$

In this section, we shall give a closed formula for the Laurent polynomial $f_{\underline{\lambda}}(v, \tilde{v})$.

We fix some notations. Let v, \tilde{v} be two indeterminates over \mathbb{Z} . Let $\tilde{\mathcal{A}} := \mathbb{Z}[v, v^{-1}, \tilde{v}, \tilde{v}^{-1}]$. The Hecke algebra $\mathcal{H}_{v, \tilde{v}}(B_n)$ of type B_n over $\tilde{\mathcal{A}}$ is the associative unital $\tilde{\mathcal{A}}$ -algebra with generators T_0, T_1, \dots, T_{n-1} subject to the following relations

$$\begin{aligned} (T_0 + 1)(T_0 - \tilde{v}) &= 0, \\ T_0 T_1 T_0 T_1 &= T_1 T_0 T_1 T_0, \\ (T_i + 1)(T_i - v) &= 0, \quad \text{for } 1 \leq i \leq n-1, \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1}, \quad \text{for } 1 \leq i \leq n-2, \\ T_i T_j &= T_j T_i, \quad \text{for } 0 \leq i < j-1 \leq n-2. \end{aligned}$$

Let tr be the trace form on the Hecke algebras $\mathcal{H}_{v, \tilde{v}}(B_n)$ and $\mathcal{H}_v(\mathfrak{S}_n)$ as defined in [31, p. 697, line 5]. Note that the trace form is denoted by τ in [31, p. 697, line -5].

Definition 4.1. (See [13, (2.1)], [8, (3.8)].) For any nonnegative integers k, a, b , we set

$$\begin{aligned} u_k^+ &= \begin{cases} 1, & \text{if } k = 0, \\ \prod_{i=1}^k (v^{i-1} + T_{i-1} \cdots T_1 T_0 T_1 \cdots T_{i-1}), & \text{if } 1 \leq k \leq n, \end{cases} \\ u_k^- &= \begin{cases} 1, & \text{if } k = 0, \\ \prod_{i=1}^k (\tilde{v} v^{i-1} - T_{i-1} \cdots T_1 T_0 T_1 \cdots T_{i-1}), & \text{if } 1 \leq k \leq n, \end{cases} \\ h_{a,b} &= T_{w_{a,b}} \end{aligned}$$

where

$$w_{a,b} = \begin{cases} \underbrace{s_a \cdots s_1}_{a} \underbrace{s_{a+1} \cdots s_2}_{b} \cdots \underbrace{s_{a+b-1} \cdots s_b}_{b}, & \text{if } a, b \text{ are positive integers,} \\ 1, & \text{if } a \text{ or } b \text{ is zero.} \end{cases}$$

By some abuse of notations, for each integer $i \in \{1, 2, \dots, n-1\} \setminus \{a\}$, we set

$$\hat{i} = \begin{cases} i + n - a, & \text{if } 1 \leq i < a, \\ i - a, & \text{if } a < i \leq n - 1. \end{cases}$$

Let $\hat{}$ be the group isomorphism from the Young subgroup $\mathfrak{S}_{(a, n-a)}$ onto the Young subgroup $\mathfrak{S}_{(n-a, a)}$ which is defined on generators by $\hat{s}_i = s_{\hat{i}}$ for each integer $i \in \{1, 2, \dots, n-1\} \setminus \{a\}$. We use the same notation $\hat{}$ to denote the algebra isomorphism from the Hecke algebra $\mathcal{H}_v(\mathfrak{S}_{(a, n-a)})$ onto the Hecke algebra $\mathcal{H}_v(\mathfrak{S}_{(n-a, a)})$ which is defined on generators by $\hat{T}_i = T_{\hat{i}}$ for each integer $i \in \{1, 2, \dots, n-1\} \setminus \{a\}$. By abuse of notation, the inverse of $\hat{}$ will also be denoted by $\hat{}$. By [9, (3.23)], we know that there exists an element $z_{a, n-a}$ in the center of $\mathcal{H}_v(\mathfrak{S}_{(a, n-a)}) \otimes_{\mathcal{A}} \tilde{\mathcal{A}}$ such that

$$u_{n-a}^- h_{n-a, a} u_a^+ h_{a, n-a} u_{n-a}^- h_{n-a, a} u_a^+ = u_{n-a}^- h_{n-a, a} u_a^+ z_{a, n-a}. \quad (4.2)$$

Let $*$ be the anti-automorphism of $\mathcal{H}_{v, \tilde{v}}(B_n)$ which is defined on generators by $T_i^* = T_i$ for each $0 \leq i \leq n-1$.

Lemma 4.3. For any integer a with $1 \leq a \leq n$, we have that

$$(4.3.1) \quad u_a^+ h_{a, n-a} u_{n-a}^- h_{n-a, a} u_a^+ h_{a, n-a} u_{n-a}^- = u_a^+ h_{a, n-a} u_{n-a}^- \hat{z}_{a, n-a},$$

$$(4.3.2) \quad (z_{a, n-a})^* = z_{a, n-a}.$$

Proof. Using the same argument as [9, (3.23)], we know that there exists an element $z'_{n-a, a}$ in the center of $\mathcal{H}_q(\mathfrak{S}_{(n-a, a)}) \otimes_{\mathcal{A}} \tilde{\mathcal{A}}$ such that

$$u_a^+ h_{a, n-a} u_{n-a}^- h_{n-a, a} u_a^+ h_{a, n-a} u_{n-a}^- = u_a^+ h_{a, n-a} u_{n-a}^- z'_{n-a, a}.$$

Note that

$$\begin{aligned} u_{n-a}^- h_{n-a, a} u_a^+ h_{a, n-a} u_{n-a}^- h_{n-a, a} u_a^+ h_{a, n-a} u_{n-a}^- h_{n-a, a} u_a^+ \\ = u_{n-a}^- h_{n-a, a} u_a^+ h_{a, n-a} u_{n-a}^- h_{n-a, a} u_a^+ z_{a, n-a} = u_{n-a}^- h_{n-a, a} u_a^+ z_{a, n-a}^2. \end{aligned}$$

On the other hand, we have that

$$\begin{aligned} u_{n-a}^- h_{n-a, a} u_a^+ h_{a, n-a} u_{n-a}^- h_{n-a, a} u_a^+ h_{a, n-a} u_{n-a}^- h_{n-a, a} u_a^+ \\ = u_{n-a}^- h_{n-a, a} u_a^+ h_{a, n-a} u_{n-a}^- z'_{n-a, a} h_{n-a, a} u_a^+ = u_{n-a}^- h_{n-a, a} u_a^+ h_{a, n-a} u_{n-a}^- z'_{n-a, a} h_{n-a, a} u_a^+ \hat{z}'_{n-a, a} \\ = u_{n-a}^- h_{n-a, a} u_a^+ z_{a, n-a} \hat{z}'_{n-a, a}. \end{aligned}$$

It follows that $z_{a, n-a}^2 = z_{a, n-a} \hat{z}'_{n-a, a}$. Note that (see [9]) $z_{a, n-a}$ is invertible in $\mathcal{H}_{v, \tilde{v}}(\mathfrak{S}_n) \otimes_{\tilde{\mathcal{A}}} \mathbb{Q}(v, \tilde{v})$. Hence we conclude that $z'_{n-a, a} = \hat{z}_{a, n-a}$, as required. This proves 1).

Applying the anti-automorphism $*$ to both sides of (4.2), we get that

$$\begin{aligned} u_a^+ h_{a, n-a} u_{n-a}^- \hat{z}_{a, n-a} = u_a^+ h_{a, n-a} u_{n-a}^- h_{n-a, a} u_a^+ h_{a, n-a} u_{n-a}^- \\ = z_{a, n-a}^* u_a^+ h_{a, n-a} u_{n-a}^- = u_a^+ h_{a, n-a} u_{n-a}^- \widehat{z_{a, n-a}^*}, \end{aligned}$$

which implies that $(z_{a, n-a})^* = z_{a, n-a}$, as required. \square

For any a -bipartition $\underline{\lambda} = (\lambda^{(1)}, \lambda^{(2)})$ of n , we define $z_{\underline{\lambda}} = z_{\lambda^{(1)}} \widehat{z_{\lambda^{(2)}}}$, where $z_{\lambda^{(1)}} = x_{\lambda^{(1)}} T_{w_{\lambda^{(1)}}} y_{\lambda^{(1)'}}$, $z_{\lambda^{(2)}} = x_{\lambda^{(2)}} T_{w_{\lambda^{(2)}}} y_{\lambda^{(2)'}}$. We set $\underline{\lambda}' := (\lambda^{(1)'}, \lambda^{(2)'})$. Recall that $\widehat{\underline{\lambda}} = (\lambda^{(2)}, \lambda^{(1)})$, which is an $(n-a)$ -bipartition of n . We define the Specht modules $S^{\underline{\lambda}}$ and the twisted Specht module $\widehat{S^{\underline{\lambda}}}$ to be:

$$S^{\underline{\lambda}} := u_{n-a}^- h_{n-a,a} u_a^+ z_{\underline{\lambda}} \mathcal{H}_{v,\tilde{v}}(B_n), \quad \widehat{S^{\underline{\lambda}}} := u_a^+ h_{a,n-a} u_{n-a}^- z_{\widehat{\underline{\lambda}}} \mathcal{H}_{v,\tilde{v}}(B_n).$$

Let $\theta_{\underline{\lambda}}$ (resp., $\delta_{\underline{\lambda}}$) be the right $\mathcal{H}_{v,\tilde{v}}(B_n)$ -module homomorphism from $S^{\underline{\lambda}}$ to $\widehat{S^{\underline{\lambda}}}$ (resp., from $\widehat{S^{\underline{\lambda}}}$ to $S^{\underline{\lambda}}$) given by left multiplication with $u_a^+ h_{a,n-a}$ (resp., with $u_{n-a}^- h_{n-a,a}$). It is clear that both $\theta_{\underline{\lambda}}$ and $\delta_{\underline{\lambda}}$ are well defined right $\mathcal{H}_{v,\tilde{v}}(B_n)$ -module homomorphisms.

Lemma 4.4. *For any a -bipartition $\underline{\lambda}$ of n , there exists a Laurent polynomial $f_{\underline{\lambda}}(v, \tilde{v}) \in \widetilde{\mathcal{A}}$ such that $z_{a,n-a} z_{\underline{\lambda}} = f_{\underline{\lambda}}(v, \tilde{v}) z_{\widehat{\underline{\lambda}}}$, $\widehat{z_{n-a,a}} z_{\widehat{\underline{\lambda}}} = f_{\underline{\lambda}}(v, \tilde{v}) z_{\underline{\lambda}}$. In particular, both $\theta_{\underline{\lambda}} \delta_{\underline{\lambda}}$ and $\delta_{\underline{\lambda}} \theta_{\underline{\lambda}}$ are scalar multiplication by $f_{\underline{\lambda}}(v, \tilde{v})$.*

Proof. The first part is an easy consequence of [7, (4.1)], by using the same argument as in the proof of [19, (3.2)]. The second part follows from Lemma 4.3. \square

Let μ be a partition of a . Let $z_{\mu} := x_{\mu} T_{w_{\mu}} y_{\mu'}$. Then $S^{\mu} := z_{\mu} \mathcal{H}_v(\mathfrak{S}_a)$ is the Specht module of $\mathcal{H}_v(\mathfrak{S}_a)$ corresponding to μ . By [8, (3.5)] and [32, (5.2), (5.3)], the dual Specht module S_{μ} of $\mathcal{H}_v(\mathfrak{S}_a)$ is isomorphic to the right ideal generated by $y_{\mu'} T_{w_{\mu'}} x_{\mu}$. Let θ_{μ} (resp., δ_{μ}) be the right $\mathcal{H}_v(\mathfrak{S}_a)$ -module homomorphism from S^{μ} to $y_{\mu'} T_{w_{\mu'}} x_{\mu} \mathcal{H}_v(\mathfrak{S}_a)$ (resp., from $y_{\mu'} T_{w_{\mu'}} x_{\mu} \mathcal{H}_v(\mathfrak{S}_a)$ to S^{μ}) given by left multiplication with $y_{\mu'} T_{w_{\mu'}}$ (resp., with $x_{\mu} T_{w_{\mu}}$). The following result is coming from [29, (5.8)]. But we shall give a different proof here in order to illustrate the technique which will be used in the proof of the main theorem in this section.

Lemma 4.5. (See [29, (5.8)].) *Let μ be a partition of a . Then both $\theta_{\mu} \delta_{\mu}$ and $\delta_{\mu} \theta_{\mu}$ are scale multiplication of $v^{\ell(w_{\mu})} s_{\mu}(v)$, where $s_{\mu}(v) \in \mathcal{A}$ is the Schur element associated to μ . In particular,*

$$z_{\mu} T_{w_{\mu'}} z_{\mu} = v^{\ell(w_{\mu})} s_{\mu}(v) z_{\mu}.$$

Proof. By [7, (4.1)], we have that

$$x_{\mu} (T_{w_{\mu}} y_{\mu'} T_{w_{\mu'}} x_{\mu} T_{w_{\mu}}) y_{\mu'} = r_{\mu}(v) x_{\mu} T_{w_{\mu}} y_{\mu'},$$

for some $r_{\mu}(v) \in \mathcal{A}$. Therefore, for any $h \in \mathcal{H}_v(\mathfrak{S}_a)$,

$$\begin{aligned} \delta_{\mu} \theta_{\mu}(z_{\mu} h) &= x_{\mu} T_{w_{\mu}} y_{\mu'} T_{w_{\mu'}} z_{\mu} h = z_{\mu} T_{w_{\mu'}} z_{\mu} h \\ &= x_{\mu} (T_{w_{\mu}} y_{\mu'} T_{w_{\mu'}} x_{\mu} T_{w_{\mu}}) y_{\mu'} h = r_{\mu}(v) x_{\mu} T_{w_{\mu}} y_{\mu'} h = r_{\mu}(v) z_{\mu} h, \end{aligned}$$

We claim that $r_{\mu}(v) \neq 0$. In fact, by [31, (5.9)],

$$\begin{aligned} \text{tr}(\delta_{\mu}(y_{\mu'} T_{w_{\mu'}} x_{\mu})) &= \text{tr}(x_{\mu} T_{w_{\mu}} y_{\mu'} T_{w_{\mu'}} x_{\mu}) = \text{tr}(x_{\mu}^2 T_{w_{\mu}} y_{\mu'} T_{w_{\mu'}}) \\ &= \left(\sum_{w \in \mathfrak{S}_{\mu}} v^{\ell(w)} \right) \text{tr}(x_{\mu} T_{w_{\mu}} y_{\mu'} T_{w_{\mu'}}) = v^{\ell(w_{\mu})} \sum_{w \in \mathfrak{S}_{\mu}} v^{\ell(w)} \neq 0. \end{aligned}$$

It follows that $\delta_{\mu}(y_{\mu'} T_{w_{\mu'}} x_{\mu}) \neq 0$, hence $(\delta_{\mu})_{\mathbb{Q}(v)}$ is an $\mathcal{H}_v(\mathfrak{S}_a) \otimes_{\mathcal{A}} \mathbb{Q}(v)$ -module isomorphism between two simple $(\mathcal{H}_v(\mathfrak{S}_a) \otimes_{\mathcal{A}} \mathbb{Q}(v))$ -modules. By similar argument, one can show that $(\theta_{\mu})_{\mathbb{Q}(v)}$

is an $(\mathcal{H}_v(\mathfrak{S}_a) \otimes_{\mathcal{A}} \mathbb{Q}(v))$ -module isomorphism between two simple $(\mathcal{H}_v(\mathfrak{S}_a) \otimes_{\mathcal{A}} \mathbb{Q}(v))$ -modules. In particular, this implies that $r_\mu(v) \neq 0$, as required.

It remains to show $r_\mu(v) = v^{\ell(w_\mu)} s_\mu(v)$. Since $r_\mu(v) \neq 0$, $r_\mu(v)^{-1} z_\mu T_{w_{\mu'}}$ is an idempotent in $\mathcal{H}_v(\mathfrak{S}_a) \otimes_{\mathcal{A}} \mathbb{Q}(v)$. Clearly

$$S^\mu \otimes_{\mathcal{A}} \mathbb{Q}(v) = r_\mu(v)^{-1} z_\mu T_{w_{\mu'}} (\mathcal{H}_v(\mathfrak{S}_a) \otimes_{\mathcal{A}} \mathbb{Q}(v)).$$

Applying [31, (1.6),(5.9)], we get that

$$s_\mu(v) = \frac{1}{\text{tr}(r_\mu(v)^{-1} z_\mu T_{w_{\mu'}})} = \frac{r_\mu(v)}{\text{tr}(z_\mu T_{w_{\mu'}})} = v^{-\ell(w_\mu)} r_\mu(v).$$

Clearly $\delta_\mu \theta_\mu = v^{\ell(w_\mu)} s_\mu(v)$ implies that $\theta_\mu \delta_\mu = v^{\ell(w_\mu)} s_\mu(v)$. This completes the proof of the theorem. \square

Theorem 4.6. *With the above notations, we have that*

$$f_{\underline{\lambda}}(v, \tilde{v}) = v^{\frac{n(n-1)}{2}} \tilde{v}^{n-a} \frac{s_{\underline{\lambda}}(v, \tilde{v})}{s_{\lambda^{(1)}}(v) s_{\lambda^{(2)}}(v)} \in \tilde{\mathcal{A}},$$

where $s_{\underline{\lambda}}(v, \tilde{v})$ (resp., $s_{\lambda^{(1)}}(v)$, $s_{\lambda^{(2)}}(v)$) is the Schur element associated to the bipartition $\underline{\lambda}$ (resp., the partitions $\lambda^{(1)}$, $\lambda^{(2)}$). In particular, in the ring $\tilde{\mathcal{A}}$,

$$s_{\lambda^{(1)}}(v) s_{\lambda^{(2)}}(v) |s_{\underline{\lambda}}(v, \tilde{v}).$$

Proof. For an a -bipartition $\underline{\lambda} = (\lambda^{(1)}, \lambda^{(2)})$, we define $T_{w_{\underline{\lambda}}} = T_{w_{\lambda^{(1)}}} \widehat{T_{w_{\lambda^{(2)}}}}$. Then $T_{w_{\underline{\lambda}}} = T_{w_{\lambda^{(2)}}} \widehat{T_{w_{\lambda^{(1)}}}}$. Now applying Lemma 4.5, we get that

$$\begin{aligned} z_{\underline{\lambda}} T_{w_{\underline{\lambda}}} z_{\underline{\lambda}} &= z_{\lambda^{(2)}} T_{w_{\lambda^{(2)}}} z_{\lambda^{(1)}} \widehat{z_{\lambda^{(1)}} T_{w_{\lambda^{(1)}}} z_{\lambda^{(1)}}} \\ &= v^{\ell(w_{\lambda^{(2)}}) + \ell(w_{\lambda^{(1)}})} s_{\lambda^{(2)}}(v) s_{\lambda^{(1)}}(v) z_{\lambda^{(2)}} \widehat{z_{\lambda^{(1)}}} \\ &= v^{\ell(w_{\lambda^{(2)}}) + \ell(w_{\lambda^{(1)}})} s_{\lambda^{(2)}}(v) s_{\lambda^{(1)}}(v) z_{\underline{\lambda}}, \end{aligned}$$

where $s_{\lambda^{(1)}}(v)$, $s_{\lambda^{(2)}}(v)$ are the Schur elements associated to the partitions $\lambda^{(1)}$, $\lambda^{(2)}$, respectively.

Let $A(v, \tilde{v}) := v^{-\ell(w_{\lambda^{(2)}}) - \ell(w_{\lambda^{(1)}})} f_{\underline{\lambda}}(v, \tilde{v})^{-1} s_{\lambda^{(2)}}(v)^{-1} s_{\lambda^{(1)}}(v)^{-1}$. Then we have

$$\begin{aligned} &(A(v, \tilde{v}) u_{n-a}^- h_{n-a,a} u_a^+ z_{\underline{\lambda}} h_{a,n-a} T_{w_{\underline{\lambda}}})^2 \\ &= (A(v, \tilde{v}) u_{n-a}^- h_{n-a,a} u_a^+ h_{a,n-a} z_{\underline{\lambda}} T_{w_{\underline{\lambda}}})^2 \\ &= A(v, \tilde{v})^2 u_{n-a}^- h_{n-a,a} u_a^+ h_{a,n-a} z_{\underline{\lambda}} T_{w_{\underline{\lambda}}} u_{n-a}^- h_{n-a,a} u_a^+ h_{a,n-a} z_{\underline{\lambda}} T_{w_{\underline{\lambda}}} \\ &= A(v, \tilde{v})^2 u_{n-a}^- h_{n-a,a} u_a^+ h_{a,n-a} u_{n-a}^- h_{n-a,a} u_a^+ h_{a,n-a} z_{\underline{\lambda}} T_{w_{\underline{\lambda}}} z_{\underline{\lambda}} T_{w_{\underline{\lambda}}} \\ &= A(v, \tilde{v}) f_{\underline{\lambda}}(v, \tilde{v})^{-1} u_{n-a}^- h_{n-a,a} u_a^+ h_{a,n-a} u_{n-a}^- h_{n-a,a} u_a^+ h_{a,n-a} z_{\underline{\lambda}} T_{w_{\underline{\lambda}}} \\ &= A(v, \tilde{v}) f_{\underline{\lambda}}(v, \tilde{v})^{-1} u_{n-a}^- h_{n-a,a} u_a^+ z_{a,n-a} h_{a,n-a} z_{\underline{\lambda}} T_{w_{\underline{\lambda}}} \\ &= A(v, \tilde{v}) f_{\underline{\lambda}}(v, \tilde{v})^{-1} u_{n-a}^- h_{n-a,a} u_a^+ z_{a,n-a} z_{\underline{\lambda}} h_{a,n-a} T_{w_{\underline{\lambda}}} \\ &= A(v, \tilde{v}) u_{n-a}^- h_{n-a,a} u_a^+ z_{\underline{\lambda}} h_{a,n-a} T_{w_{\underline{\lambda}}}. \end{aligned}$$

In other words, $A(v, \tilde{v})u_{n-a}^-h_{n-a,a}u_a^+z_{\underline{\lambda}}h_{a,n-a}Tw_{\hat{\underline{\lambda}}}$ is an idempotent of the Hecke algebra $\mathcal{H}_{v,\tilde{v}}(B_n) \otimes_{\tilde{\mathcal{A}}} \mathbb{Q}(v, \tilde{v})$. Since

$$S_{\underline{\lambda}} \otimes_{\tilde{\mathcal{A}}} \mathbb{Q}(v, \tilde{v}) = A(v, \tilde{v})u_{n-a}^-h_{n-a,a}u_a^+z_{\underline{\lambda}}h_{a,n-a}Tw_{\hat{\underline{\lambda}}}(\mathcal{H}_{v,\tilde{v}}(B_n) \otimes_{\tilde{\mathcal{A}}} \mathbb{Q}(v, \tilde{v})),$$

it follows from [31, (1.6)] that

$$s_{\underline{\lambda}}(v, \tilde{v}) = \frac{1}{\text{tr}(A(v, \tilde{v})u_{n-a}^-h_{n-a,a}u_a^+z_{\underline{\lambda}}h_{a,n-a}Tw_{\hat{\underline{\lambda}}})},$$

where $s_{\underline{\lambda}}(v, \tilde{v})$ is the Schur element associated to the bipartition $\underline{\lambda}$. On the other hand, if we set $q = v$, $Q_1 = -1$, $Q_2 = \tilde{v}$, and $\mu = (\lambda^{(2)}, \lambda^{(1)})$, then the element m_{μ} in [31] is

$$x_{\lambda^{(2)}}\widehat{x_{\lambda^{(1)}}}\left(\prod_{k=1}^{n-a}(-v^{1-k})\right)u_{n-a}^-$$

in our notation, and the element $n_{\mu'}$ in [31] is

$$y_{\lambda^{(1)'}}\widehat{y_{\lambda^{(2)'}}}\left(\prod_{k=1}^av^{1-k}\right)u_a^+$$

in our notation. Note also that the element $w_{\mu'}$ in [31, (5.9)] is just

$$w_{a,n-a}w_{\lambda^{(2)'}}\widehat{w_{\lambda^{(1)'}}}$$

in our notation. Therefore, the element $z_{\mu}Tw_{\mu'}$ in the notation of [31, (5.9)] is just

$$\left(\prod_{k=1}^{n-a}(-v^{1-k})\right)\left(\prod_{k=1}^av^{1-k}\right)u_{n-a}^-h_{n-a,a}u_a^+z_{\underline{\lambda}}h_{a,n-a}Tw_{\hat{\underline{\lambda}}}$$

in our notations. Applying [31, (5.9)], we have

$$\begin{aligned} & \text{tr}\left(\left(\prod_{k=1}^{n-a}(-v^{1-k})\right)\left(\prod_{k=1}^av^{1-k}\right)u_{n-a}^-h_{n-a,a}u_a^+z_{\underline{\lambda}}h_{a,n-a}Tw_{\hat{\underline{\lambda}}}\right) \\ &= (-1)^n v^{\ell(w_{a,n-a})+\ell(w_{\lambda^{(2)}})+\ell(w_{\lambda^{(1)}})} (-1)^a \tilde{v}^{n-a}. \end{aligned}$$

It follows that

$$\text{tr}(u_{n-a}^-h_{n-a,a}u_a^+z_{\underline{\lambda}}h_{a,n-a}Tw_{\hat{\underline{\lambda}}}) = v^{\frac{n(n-1)}{2}+\ell(w_{\lambda^{(2)}})+\ell(w_{\lambda^{(1)}})} \tilde{v}^{n-a}.$$

Hence,

$$f_{\underline{\lambda}}(v, \tilde{v}) = v^{\frac{n(n-1)}{2}} \tilde{v}^{n-a} \frac{s_{\underline{\lambda}}(v, \tilde{v})}{s_{\lambda^{(1)}}(v)s_{\lambda^{(2)}}(\tilde{v})} \in \tilde{\mathcal{A}},$$

as required. \square

Corollary 4.7. *With the above notations, we have that*

$$f_{\hat{\lambda}}(v) = v^{\frac{n(n-1)}{2}} \frac{s_{\hat{\lambda}}(v, 1)}{s_{\hat{\lambda}(1)}(v) s_{\hat{\lambda}(2)}(v)}.$$

5. Twining character formulae

The purpose of this final section is to derive a closed formula of the decomposition number $[\tilde{\Delta}_{\lambda, \lambda}^+ : \tilde{L}_{\mu, \mu}^+]$ (where $\lambda, \mu \in \Lambda^+(m)$) in terms of the elements $f_{(\lambda, \lambda)}(q)$, $f_{(\mu, \mu)}(q)$, and thus proving our main result Theorem 2.9.

Let M be an arbitrary finite dimensional $\tilde{A}(m)$ -module. Then $M\theta \subseteq M$. For each $\mu \in \Lambda(m)$, it follows from the definition that

$$\theta \phi_{\mu, \mu}^1 = \hat{\phi}_{\mu, \mu}^1 \theta, \quad \phi_{\mu, \mu}^1 \theta = \theta \hat{\phi}_{\mu, \mu}^1.$$

Let $M_{(\mu, \mu)}$ be the (μ, μ) -weight space of the $\tilde{A}(m)$ -module M . Note that for any $x \in M$, $x \in M_{(\mu, \mu)}$ if and only if $x\phi_{\mu, \mu}^1 = x = x\hat{\phi}_{\mu, \mu}^1$. Then, it is easy to verify that $M_{(\mu, \mu)}\theta \subseteq M_{(\mu, \mu)}$. Therefore, it makes sense to talk about the trace of θ on $M_{(\mu, \mu)}$.

We define the twining character of the $\tilde{A}(m)$ -module M as follows:

$$\text{ch}^\theta(M) := \sum_{\mu \in \Lambda(m)} \text{Tr}(\theta, M_{(\mu, \mu)}) e^\mu \in K[\Lambda(m)],$$

where $K[\Lambda(m)]$ denotes the group algebra of the additive group $\Lambda(m)$. It is easy to see that ch^θ induces a homomorphism from the Grothendieck group $\mathcal{R}(\tilde{A}(m))$ to $K[\Lambda(m)]$. We denote this homomorphism again by ch^θ . We use π to denote the natural map from the group ring $\mathbb{Z}[\Lambda(m)]$ to the group algebra $K[\Lambda(m)]$.

Lemma 5.1. *For any $\lambda, \mu \in \Lambda^+(m)$ with $\lambda \neq \mu$, we have that*

$$\begin{aligned} \text{ch}^\theta(\tilde{\Delta}_{\lambda, \lambda}^+) &= -\text{ch}^\theta(\tilde{\Delta}_{\lambda, \lambda}^-) = \sqrt{f_{(\lambda, \lambda)}(q)} \pi(\text{ch } \Delta_\lambda), \\ \text{ch}^\theta(\tilde{L}_{\lambda, \lambda}^+) &= -\text{ch}^\theta(\tilde{L}_{\lambda, \lambda}^-) = \sqrt{f_{(\lambda, \lambda)}(q)} \pi(\text{ch } L_\lambda), \\ \text{ch}^\theta(\tilde{L}_{\lambda, \mu}) &= 0, \end{aligned}$$

where $\text{ch } \Delta_\lambda$, $\text{ch } L_\lambda$ denote the formal characters of the Weyl module and irreducible module associated to λ of the q -Schur algebra $S_q(m)$, respectively.

Proof. Let $v \in \Lambda(m)$. By definition, $\tilde{L}_{\lambda, \mu} = (L_\lambda \otimes L_\mu) \oplus (L_\lambda \otimes L_\mu)\theta$, and the (v, v) -weight space of $\tilde{L}_{\lambda, \mu}$ is just

$$((L_\lambda)_v \otimes (L_\mu)_v) \oplus ((L_\lambda)_v \otimes (L_\mu)_v)\theta.$$

It is easy to see that the trace of θ on this space is 0. This proves $\text{ch}^\theta(\tilde{L}_{\lambda, \mu}) = 0$.

Let v_λ (resp., \hat{v}_λ) be the highest weight vector of Δ_λ (resp., of $\hat{\Delta}_\lambda$). By definition, we know that the (v, v) -weight space of $\tilde{\Delta}_{\lambda, \lambda}^+$ is $(\Delta_\lambda)_v \otimes (\hat{\Delta}_\lambda)_v$. Let $\{v_\lambda x_i\}_{i=1}^k$ be a K -basis of $(\Delta_\lambda)_v$, then $\{\hat{v}_\lambda \hat{x}_i\}_{i=1}^k$ is a K -basis of $(\hat{\Delta}_\lambda)_v$. We now apply Corollary 3.17. For any integer $1 \leq i, j \leq k$, we have

$$(v_\lambda x_i \otimes \hat{v}_\lambda \hat{x}_j) \frac{\theta}{\sqrt{f_{(\lambda, \lambda)}(q)}} = v_\lambda x_j \otimes \hat{v}_\lambda \hat{x}_i.$$

It follows that

$$\mathrm{Tr}(\theta, (\tilde{\Delta}_{\lambda,\lambda}^+)_{(v,v)}) = \sqrt{f_{(\lambda,\lambda)}(q)} \dim(\Delta_\lambda)_v.$$

Hence $\mathrm{ch}^\theta(\tilde{\Delta}_{\lambda,\lambda}^+) = \sqrt{f_{(\lambda,\lambda)}(q)} \pi(\mathrm{ch} \Delta_\lambda)$. The remaining equalities can be proved in a similar way. This completes the proof of the lemma. \square

For any $\lambda, \mu, v \in \Lambda^+(m)$ with $\mu \neq v$, we set

$$d_{\lambda,\mu} := [\Delta_\lambda : L_\mu], \quad m_{\lambda,\mu} := [\tilde{\Delta}_{\lambda,\lambda}^+ : \tilde{L}_{\mu,\mu}^+], \quad m_{\mu,v}^\lambda := [\tilde{\Delta}_{\lambda,\lambda}^+ : \tilde{L}_{\mu,v}].$$

By (3.18), it is clear that

$$[\tilde{\Delta}_{\lambda,\lambda}^+ : \tilde{L}_{\mu,\mu}^-] = d_{\lambda,\mu}^2 - m_{\lambda,\mu}.$$

Therefore, we have the following equality in the Grothendieck group $\mathcal{R}(\tilde{A}(m))$:

$$[\tilde{\Delta}_{\lambda,\lambda}^+] = \sum_{\mu \in \Lambda^+(m)} (m_{\lambda,\mu} [\tilde{L}_{\mu,\mu}^+] + (d_{\lambda,\mu}^2 - m_{\lambda,\mu}) [\tilde{L}_{\mu,\mu}^-]) + \sum_{\substack{\mu, v \in \Lambda^+(m) \\ \mu \neq v}} m_{\mu,v}^\lambda [\tilde{L}_{\mu,v}].$$

We apply the map ch^θ to the above equality and using Lemma 5.1. It follows that

$$\sqrt{f_{(\lambda,\lambda)}(q)} \pi(\mathrm{ch} \Delta_\lambda) = \sum_{\mu \in \Lambda^+(m)} (2m_{\lambda,\mu} - d_{\lambda,\mu}^2) \sqrt{f_{(\mu,\mu)}(q)} \pi(\mathrm{ch} L_\mu). \quad (5.2)$$

On the other hand, we have that (in $\mathbb{Z}[\Lambda^+(m)]$)

$$\mathrm{ch} \Delta_\lambda = \sum_{\mu \in \Lambda^+(m)} d_{\lambda,\mu} \mathrm{ch} L_\mu,$$

and hence (in $K[\Lambda^+(m)]$)

$$\pi(\mathrm{ch} \Delta_\lambda) = \sum_{\mu \in \Lambda^+(m)} d_{\lambda,\mu} \pi(\mathrm{ch} L_\mu). \quad (5.3)$$

By the results in [5] and [15], $S_q(m)$ is an epimorphic image of the quantum algebra $U_K(\mathfrak{gl}_m)$, hence each irreducible $S_q(m)$ -module L_μ is a highest weight module over the quantum algebra $U_K(\mathfrak{gl}_m)$. It follows easily that the elements in the set $\{\pi(\mathrm{ch} L_\mu)\}$ are K -linear independent. Therefore, we can compare the coefficients of the equalities (5.2) and (5.3) and deduce that (in K)

$$m_{\lambda,\mu} = d_{\lambda,\mu} \left(\frac{\sqrt{f_{(\lambda,\lambda)}(q)}}{\sqrt{f_{(\mu,\mu)}(q)}} + d_{\lambda,\mu} \right) / 2.$$

If μ is e -restricted, we then apply Corollary 3.21 and this proves Theorem 2.9.

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